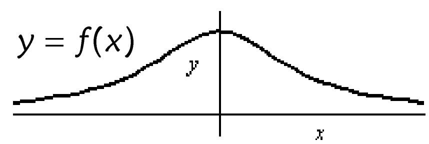
New York State Mathematics Association of Two-Year Colleges

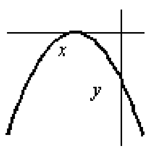
Math League Contest – Fall 2009

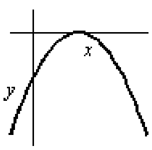
1. $x \otimes y$ is defined as: $x \otimes y = \begin{cases} xy - y^2, & y < x \\ (y+1) \otimes (x-1), & y \geq x \end{cases}$. What is the value of $(-1 \otimes 0) \otimes (4 \otimes 4)$?

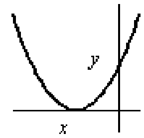
2. The graph of $y = f(x)$ is shown.

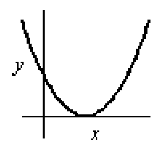


Which could be the sketch of $y = 1 - \dfrac{1}{f(x-1)}$?

a) 

b) 

c) 

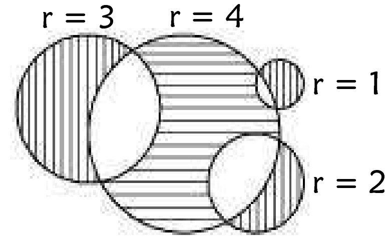
d) 

3. George and Ray get together once a week to play pool. They always play 4 games. From past experience George wins 2 of the 4 games just as often as he wins 3 of the 4 games. If George does not always win or always lose, then what is probability George wins any one game, assuming it is a constant?

4. If $\log_b(x) = y$, then $\log_{\left(\frac{1}{b}\right)}\left(\dfrac{1}{x}\right) = ?$

a) $-y$ b) y c) $\log_b(y)$ d) $\log_y(b)$

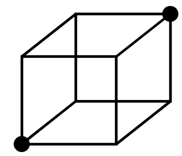
5. Let A_1 equal the total area of the vertically shaded regions, and A_2 equal the area of the horizontally shaded region. The radius of each circle is given. What is $A_2 - A_1$?



a) π b) 2π c) 3π d) cannot be determined

6. A lattice point is a point with integer coordinates. If a line segment is drawn from the point $(6, 17)$ to $(339, 794)$, how many lattice points will fall on the line <u>between</u> the endpoints?

7. A bug is confined to the inside of a cube with an edge of 1 unit in length. What is the shortest distance it can crawl to get from one corner of the cube to the opposite corner? (The bug cannot jump or fly, it must always be in contact with the cube.)



a) $\sqrt{3}$ b) 2 c) $\sqrt{5}$ d) $1+\sqrt{2}$

8. What is the radius of the circle centered at $(0,1)$ that is tangent to the graph of the parabola $y = x^2$ at exactly 2 points?

9. At a gathering of n people, everyone shakes hands with everyone else exactly once. If there were a total of 120 handshakes, what is n?

10. What is the *exact* value of $\sin(18^\circ)\sin(54^\circ)$?

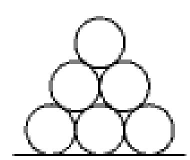
a) $\frac{1}{4}$ b) $\frac{1}{5}$ c) $\frac{1}{6}$ d) $\frac{1}{7}$

11. Given a horizontal line and two points, A - above the line and B - below the line, which of the following statements are *always* true?

I. There is a point, C, on the line such that triangle ABC is a right triangle.
II. There is a point, C, on the line that is equidistant from points A and B.
III. There is a point, C, on the line such that a circle can be drawn through A, B, and C with diameter AB.

a) I only b) III only c) I and III only d) I, II and III

12. The following diagram shows six circles each with a diameter of 1 that are *stacked* on top of each other. What is the height of figure formed?



13. What is the area *enclosed* by the graph of $x^2y^2 - 9x^2 - 25y^2 + 225 = 0$?

14. While checking the batting average of my favorite baseball player, Mickey Mantell, I noticed that if he had twice the number of hits, his average would be $\frac{7}{5}$ times as great as if his at-bats were reduced by the number of his hits. What is his batting average? Note: A batting average is determined by the number of hits divided by the number of at-bats, and assume his average is non-zero!

15. The rails of a railroad are 30 feet long. As a train passes over a point where the rails are joined, a passenger in the train can hear an audible click. The speed of the train in miles per hour is approximately the number of clicks heard in how many seconds? Select the best answer!

Note: 1 mile = 5280 feet.

a) 15 b) 20 c) 25 d) 30

16. If you add the age of a man to the age of his wife, the result is 91. He is now twice as old as she was when he was as old as she is now. How old is the man now?

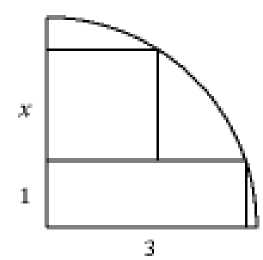
17. What is the *exact* value of $\arccos[\cos(10)]$? Note: The angle is 10 *radians*!

a) 10 b) $4\pi - 10$ c) $10 - 3\pi$ d) $3\pi - 10$

18. If $2^x = 3$ and $8^y = 18$ then y equals

a) x b) $\frac{3}{2}x$ c) $1 + \frac{1}{3}x$ d) $\frac{1}{3}(1 + 2x)$

19. A rectangle measuring 1 by 3 units is inscribed in a ¼-circle, then a square is inscribed in the remaining section as shown. What is the length, x, of each side of the inscribed square?



20. Four children, Emad, Ken, Sue, and Tim, were playing when suddenly I heard a loud noise followed by much shouting. I rushed in to find a valuable lamp had been knocked off the table and broken beyond repair. Emad and Ken spoke almost at the same time:

Emad saying, "It wasn't me!"
Ken saying, "It was Tim!"
Sue yelled, "No, it was Ken!"
Tim said, with a straight face, "Ken's a liar."

Only one of them was telling the truth, so who knocked over the lamp?

NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest - Fall 2009 - Solutions

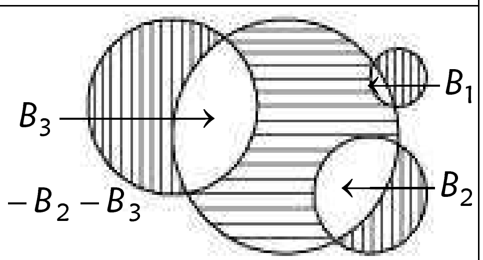
1. $-1 \otimes 0 = 1 \otimes -2 = (1)(-2) - (-2)^2 = -6$ and $4 \otimes 4 = 5 \otimes 3 = (5)(3) - 3^2 = 6$
Thus, we get $-6 \otimes 6 = 7 \otimes -7 = (7)(-7) - (-7)^2 = -98$ **Answer: -98**

2. $y = f(x)$ has a maximum value, which is greater than 0, at $x = 0$, so $\frac{1}{f(x-1)}$ will have a minimum value, which is greater than 0, at $x = 1$. Thus, $-\frac{1}{f(x-1)}$ has a maximum value at $x = 1$, which is less than 0, and the "+1" shifts the curve up. **Answer: b**

3. Let p = the probability that George wins any one game, $1-p$ = the probability Ray wins any one game. The probability George wins 2 out of 4 games is ${}_4C_2 \cdot p^2(1-p)^2 = 6p^2(1-p)^2$. The probability George wins 3 out of 4 games is ${}_4C_3 \cdot p^3(1-p)^1 = 4p^3(1-p)$. Equating these two expressions and solving for p yields: $6p^2(1-p)^2 = 4p^3(1-p) \Rightarrow 3(1-p) = 2p$ (after dividing by p^2 and $(1-p)$ since neither is zero) $3(1-p) = 2p \Rightarrow p = \frac{3}{5}$. **Answer: $\frac{3}{5}$ or 0.6**

4. $\log_b(x) = y \Rightarrow b^y = x \Rightarrow \left(b^y\right)^{-1} = x^{-1} \Rightarrow b^{-y} = x^{-1} \Rightarrow \left(\frac{1}{b}\right)^y = \frac{1}{x} \Rightarrow \log_{\left(\frac{1}{b}\right)}\left(\frac{1}{b}\right)^y = \log_{\left(\frac{1}{b}\right)}\left(\frac{1}{x}\right)$
$\Rightarrow y = \log_{\left(\frac{1}{b}\right)}\left(\frac{1}{x}\right)$ **Answer: b**

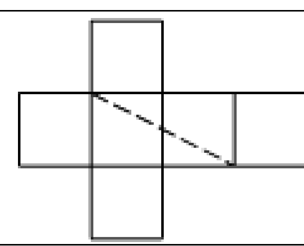
5. Let B_1 = the area shared by the circles of radius 1 and 4,
B_2 = the area shared by the circles of radius 2 and 4, and
B_3 = the area shared by the circles of radius 3 and 4 – as illustrated.

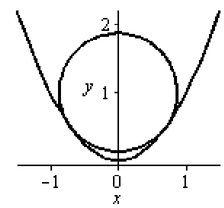
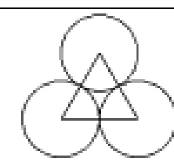
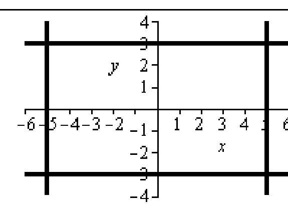


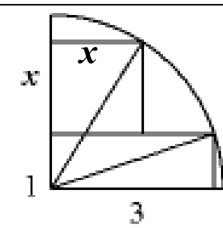
Thus, $A_1 = \left(\pi \cdot 1^2 - B_1\right) + \left(\pi \cdot 2^2 - B_2\right) + \left(\pi \cdot 3^2 - B_3\right) = 14\pi - B_1 - B_2 - B_3$
and $A_2 = \pi \cdot 4^2 - B_1 - B_2 - B_3 = 16\pi - B_1 - B_2 - B_3$
$\Rightarrow A_2 - A_1 = \left(16\pi - B_1 - B_2 - B_3\right) - \left(14\pi - B_1 - B_2 - B_3\right) = 2\pi$ **Answer: b**

6. The slope between the two points $(6, 17)$ and $(339, 794)$ is $\frac{794-17}{339-6} = \frac{777}{333} = \frac{7}{3}$. Thus, the next lattice points after $(6, 17)$ will be $(6+3, 17+7)$, $(6+2\cdot 3, 17+2\cdot 7)$, and so on. We stop when we get to the point $(339-3, 794-7) = (336, 784)$. Since $336 - 6 = 330$, we get $\frac{330}{3} = 110$ lattice points between $(6, 17)$ and $(339, 794)$. **Answer: 110**

7. Unfolding the cube gives the figure shown. The dashed line shows the shortest distance the bug can traverse to get from one corner of the cube to the opposite one. This is the hypotenuse of the right triangle with legs of length 1 and 2. Thus, the shortest distance is $\sqrt{5}$. **Answer: c**



8.	Let r represent the radius of the circle. The equation of the circle is then given by $x^2+(y-1)^2=r^2$. Substituting $y=x^2$ into the equation for the circle yields: $x^2+(x^2-1)^2=r^2 \Rightarrow x^2+x^4-2x^2+1=r^2 \Rightarrow x^4-x^2+1-r^2=0.$  The quadratic formula gives $x^2=\frac{1\pm\sqrt{1-4(1-r^2)}}{2}=\frac{1\pm\sqrt{4r^2-3}}{2}$. Since we want exactly two solutions, i.e. two points of intersection, we need x^2 to equal only one positive value, thus setting $4r^2-3=0$ yields the smallest such value of $r \Rightarrow r^2=\frac{3}{4}$. Taking the positive root gives $r=\frac{\sqrt{3}}{2}$. Answer: $r=\frac{\sqrt{3}}{2}$
9.	Each of the n people shakes hands with the $n-1$ other people for a total of $n(n-1)$ handshakes. However, this counts person A shaking hands with person B and person B shaking hands with person A as 2 different events. Thus, counting only unique handshakes we get $\frac{1}{2}n(n-1)$. Now we solve $\frac{1}{2}n(n-1)=120$ for n. $\frac{1}{2}n(n-1)=120 \Rightarrow n(n-1)=240$, hence we need two consecutive positive integers whose product is 240 $\Rightarrow 16\cdot 15=240 \Rightarrow$ Answer: $n=16$
10.	Let $x=\sin(18°)\sin(54°)$, multiply by $\cos(18°)$ and recall $\sin(2A)=2\cos(A)\sin(A)$ $\Rightarrow \cos(18°)x=\cos(18°)\sin(18°)\sin(54°)$, but $\sin(36°)=2\cos(18°)\sin(18°)$ $\Rightarrow \cos(18°)x=\frac{1}{2}\sin(36°)\sin(54°)$, and $\sin(54°)=\cos(36°)$ $\Rightarrow \cos(18°)x=\frac{1}{2}\sin(36°)\cos(36°)$, with $\cos(36°)\sin(36°)=\frac{1}{2}\sin(72°)$ $\Rightarrow \cos(18°)x=\frac{1}{2}\cdot\frac{1}{2}\sin(72°)$, and $\sin(72°)=\cos(18°)$ $\Rightarrow \cos(18°)x=\frac{1}{4}\cos(18°)$, dividing by $\cos(18°)$ gives $x=\frac{1}{4}$ Answer: a
11.	Statement II will be false when the two points are not equidistant from the line and the segment connecting them is perpendicular to the given line. Answer: c
12.	The centers of any three mutually tangent circles form an equilateral triangle with sides of length 1. The height of the equilateral triangle is $\frac{\sqrt{3}}{2}$. Thus, the height of the *stack* of 6 circles is $\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{1}{2}+\frac{1}{2}=\sqrt{3}+1$. Answer: $\sqrt{3}+1$ 
13.	Factoring: $x^2y^2-9x^2-25y^2+225=0 \Rightarrow x^2\left(y^2-9\right)-25\left(y^2-9\right)=0$ $\Rightarrow \left(x^2-25\right)\left(y^2-9\right)=0 \Rightarrow (x+5)(x-5)(y+3)(y-3)=0$ $\Rightarrow x=\pm 5, y=\pm 3$ Thus, the graph consists of the two vertical lines $x=\pm 5$ and the two horizontal lines $y=\pm 3$, which encloses a rectangle that is 10 units wide by 6 units in height. Area = 60 
14.	Let $x=$ the number of hits achieved, and $y=$ the number of at-bats. Thus, his batting average is $\frac{x}{y}$. If he had 2 times the hits, his batting average would be $\frac{2x}{y}$, which is $\frac{7}{5}$ times his average if his at-bats were only $y-x$. This gives: $\frac{2x}{y}=\frac{x}{y-x}\cdot\frac{7}{5} \Rightarrow 2\left(\frac{x}{y}\right)=\frac{\frac{7}{5}\left(\frac{x}{y}\right)}{1-\left(\frac{x}{y}\right)} \Rightarrow$

	let $A=\frac{x}{y}$ (i.e. the batting average) $\Rightarrow 2A=\frac{1.4A}{1-A}\Rightarrow 2A-2A^2=1.4A$ $\Rightarrow 0.6A-2A^2=0\Rightarrow A(0.6-2A)=0\Rightarrow A=0$ or $A=0.3$, taking the non-zero result... Answer: 0.3 or $\frac{3}{10}$
15.	Let the speed of the train be x miles/hour, then in feet/second it is $\frac{5280x}{3600}$ (since there are 5280 feet/mile and 3600 seconds/hour). There will be an audible "click" every 30 feet traversed by the train, thus $\left(\frac{5280x}{3600}\text{ feet/sec}\right)\div(30\text{ feet/click})$ gives the number of clicks/sec at speed x miles/hour. This yields $\frac{5280x}{3600\cdot 30}$ clicks/sec $=\frac{5{,}280x}{108{,}000}$ clicks/sec $\approx\frac{5{,}000x}{100{,}000}$ clicks/sec $=\frac{x}{20}$ clicks/sec $=x$ clicks/(20 sec). Hence, the number of "clicks" in 20 seconds is the approximate speed in miles/hour. Answer: b
16.	Let x = the man's age now, and y = his wife's age now. The first equation is clearly $x+y=91$. He was her age $x-y$ years ago, so when the husband was her age she was $y-(x-y)=2y-x$ years old. Hence, the second equation is $x=2(2y-x)\Rightarrow 3x=4y\Rightarrow y=\frac{3}{4}x$. Substituting that into the first equation gives: $x+\frac{3}{4}x=91\Rightarrow\frac{7}{4}x=91\Rightarrow x=52$. Answer: 52 years old
17.	Since $0\le\arccos(x)\le\pi$, we know $\arccos[\cos(10)]\ne 10$. We need to know in which quadrant 10 radians falls and it's reference angle. Since $2\pi\approx 6.28$ and $3\pi\approx 9.42$, 10 radians must be in the 3rd quadrant, with a reference angle of $10-3\pi$. However, the arccosine function yields an angle in the 2nd quadrant if the cosine is negative. Hence, $\arccos[\cos(10)]=\pi-(10-3\pi)=4\pi-10$. Answer: b
18.	$2^x=3\Rightarrow x=\log_2(3)$, $8^y=18\Rightarrow\left(2^3\right)^y=18\Rightarrow 2^{3y}=18\Rightarrow 3y=\log_2(18)$ $\Rightarrow 3y=\log_2\left(2\cdot 3^2\right)\Rightarrow 3y=\log_2(2)+\log_2\left(3^2\right)\Rightarrow 3y=1+2\log_2(3)$ $\Rightarrow 3y=1+2x\Rightarrow y=\frac{1}{3}+\frac{2}{3}x\Rightarrow$ Answer: d
19.	The radius of the circle is a diagonal of the rectangle, the Pythagorean Theorem gives $r^2=1^2+3^2\Rightarrow r=\sqrt{10}$. The Pythagorean Theorem is used again for the triangle formed by the square, rectangle, and radius: $x^2+(x+1)^2=\left(\sqrt{10}\right)^2$  $2x^2+2x+1=10\Rightarrow 2x^2+2x-9=0\Rightarrow x=\frac{-2\pm\sqrt{2^2-4\cdot 2\cdot(-9)}}{2\cdot 2}$ $\Rightarrow x=\frac{-2\pm\sqrt{76}}{4}=\frac{-1\pm\sqrt{19}}{2}\Rightarrow$ $x=\frac{\sqrt{19}-1}{2}$ (taking the positive result)
20.	Suppose Emad is the only truthful child, then Ken would be a liar and Tim would be truthful. X Suppose Ken is the only truthful child, then it was Tim and Emad would be truthful. X Suppose Sue is the only truthful child, then it was Ken and Emad would also be truthful. X Suppose Tim is the only truthful child, then it was not Tim and not Ken and must be Emad! √ Answer: Emad