New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2012

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

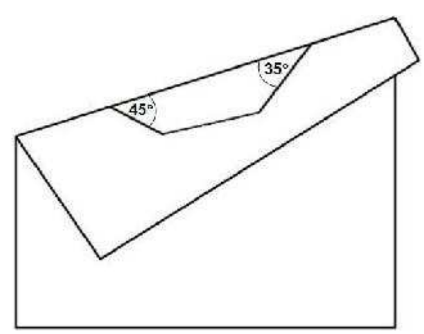
1. If $f(2x-1)=4x^2+1$, then what is $f(x)$?

2. The pattern in the expression $\sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\ldots}}}}$ continues indefinitely. To what value does it converge?
   a) $\sqrt{3}$ b) 2 c) $\sqrt{5}$ d) $1+\sqrt{2}$

3. If x and y are the smallest positive integers such that $7000x$ is a perfect square and $7000y$ is a perfect cube, then what is $x+y$?

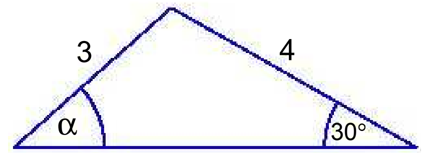
4. My video camera includes a battery pack that, when fully charged, allows one to record for up to two hours *or* playback for up to three hours. What is the maximum time, in minutes, I can record *and then* playback the entire recording on a single fully charged battery pack?

5. A regular polygon (i.e. a polygon with congruent sides and interior angles) is drawn on the back side of a piece of paper. The page is folded so that only part of the polygon is visible from the front forming angles of 45° and 35° with the fold, as shown. How many sides has the polygon?



6. How many solutions are there to the equation $x|x|=2012x+1$? $|x|$ is the absolute value of x.
   a) 1 b) 2 c) 3 d) 4

7. The diagram shows a triangle (not necessarily drawn to scale) with sides of length 3 and 4, and an angle of 30°, as shown. What is $\cos(\alpha)$?



a) $\dfrac{3}{\sqrt{5+2\sqrt{3}}}$ b) $\dfrac{3}{5}$ c) $\dfrac{\sqrt{5}}{3}$ d) $\dfrac{\sqrt{3}}{2}$

8. The minimum of seven integers is 12, the median is 24, the maximum is 30, and the mode is 27. If the average is also an integer, then which of the following could *not* be the average?
a) 20 b) 21 c) 22 d) 23

9. Two spheres, with radii 3 and 4 units, have centers that are 5 units apart. What is the radius of the circle of intersection?

10. Suppose we have two 6-sided dice, i.e. two cubes numbered 1 through 6 (one number per side), except that the numbers on each face are removable. If all the numbers are removed from both dice then randomly placed back on the faces of the dice, what is the probability a sum of 7 will then be rolled?

a) $\dfrac{5}{36}$ b) $\dfrac{1}{6}$ c) $\dfrac{2}{11}$ d) $\dfrac{3}{11}$

11. Alphametic puzzles are arithmetic problems which involve words where each letter represents a unique digit that makes the equation true. What 5-digit number is represented by **OBAMA** that satisfies the following alphametic puzzle? Hint: **Y** = 2.

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  OBAMA
+ OBAMA
 ROMNEY
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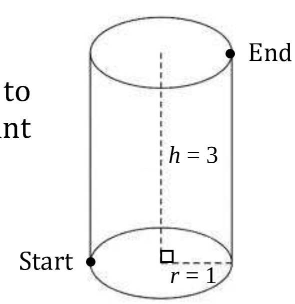
12. An ant is at the bottom of a cylinder of radius 1 and height 3 and wants to get to the top on the opposite "side" as shown. What is the minimum distance the ant can walk to achieve his goal?



13. If $\cos^4(x)-\sin^4(x)=\dfrac{1}{2}$, then what is the value of $\tan^4(x)$?

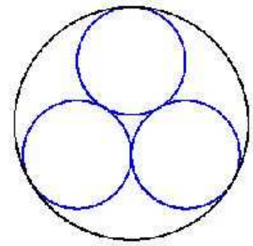
a) $\dfrac{1}{9}$ b) $\dfrac{1}{4}$ c) 1 d) NOTA

14. If x and y are two different positive integers such that $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{5}$, what is $x+y$?

15. I have one dollar in change consisting only of pennies, nickels, and dimes. If I have a total of 45 coins, then how many are nickels?

16. The equation $\log_2(x) + \log_x(2) = 4$ has two real solutions. What is the product of these solutions? Only a *completely* simplified answer will be given credit!

17. Let $f(x) = 2x^4 - x^3 - 22x^2 + 51x - 20$. Given that $f(2-i) = 0$, what is the sum of all the real roots of $f(x)$?



18. A circle of radius one has three smaller circles, of equal radii, inscribed within it that are mutually tangent to each other and the larger circle, as shown. What is the radius of each of the three smaller circles?

a) $\dfrac{5}{12}$ b) $\dfrac{\sqrt{3}}{4}$ c) $\dfrac{\sqrt{3}+1}{6}$ d) $2\sqrt{3} - 3$

19. Alice, Bob, Cathy, David, and Erin must be seated on one side of a rectangular table. However, Bob and Erin do not get along. How many different seating arrangements are there if Bob and Erin cannot sit next to each other?

a) 24 b) 48 c) 72 d) 96

20. Four logicians at a restaurant have just finished their lunch when the waitress comes by and asks, "Do you all want coffee?" The first logician answers, "I don't know." The second and third logicians also answer, "I don't know." The fourth logician answers, "No." Assuming all four of them responded with correct answers, how many want coffee?

a) 0 b) 1 c) 3 d) It cannot be determined from the given information.

NYSMATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2012 ~ Solutions

1. We need to find the inverse function of $g(x)=2x-1$ so we can replace the x in $f(2x-1)$ to obtain $f(x)$. Since $g^{-1}(x)=\frac{1}{2}(x+1)$, $f\left(2\cdot\frac{1}{2}(x+1)-1\right)=4\left(\frac{1}{2}(x+1)\right)^2+1$. This gives $f(x)=x^2+2x+2$.

   Answer: $f(x)=x^2+2x+2$

2. Let $x=\sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\ldots}}}}\Rightarrow x^2=1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\ldots}}}=1+2x$ Thus, $x^2=1+2x$, now solve for x. $x^2-2x-1=0\Rightarrow x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}=\dfrac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$. However, since $x>0$, $x=1+\sqrt{2}$.

   Answer: d

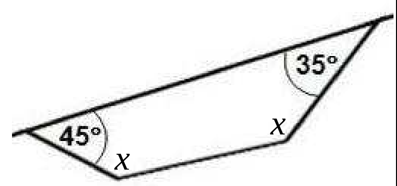
3. Since $7000=7\cdot10^3=7\cdot2^3\cdot5^3$, $x=7\cdot2\cdot5=70$ so that $7000x$ is a perfect square (i.e. all prime factors have an exponent that is a multiple of 2), and $y=7^2=49$ so that $7000x$ is a perfect cube (i.e. all prime factors have an exponent that is a multiple of 3). Hence, $x+y=119$.

   Answer: 119

4. Let x = the maximum time (in hours) I can record video. Thus, $\dfrac{x}{2}$ fraction of the battery power will be consumed. This leaves $\dfrac{2-x}{2}$ fraction of battery power remaining for playback (out of 3 hours). Hence, we require $x=3\left(\dfrac{2-x}{2}\right)\Rightarrow x=\dfrac{6}{5}=1.2$ hours = 72 minutes.

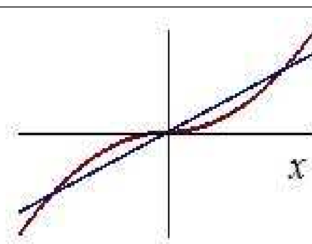
   Answer: 72

5. Let x = the angle measure of each of the unknown angles of the quadrilateral formed. Since the sum of the interior angles of all quadrilaterals is 360°, we get 45°+35°+2x=360° or x = 140°. The sum of the interior angles of an n-sided polygon is $180^\circ(n-2)$, which means that each interior angle of a regular n-sided polygon is $\dfrac{180^\circ(n-2)}{n}$. Thus, $\dfrac{180^\circ(n-2)}{n}=140^\circ\Rightarrow180n-360=140n\Rightarrow n=9$.

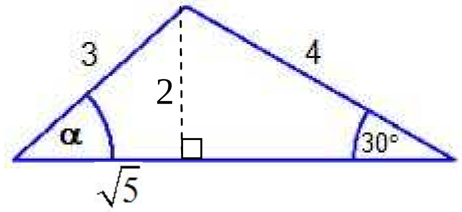
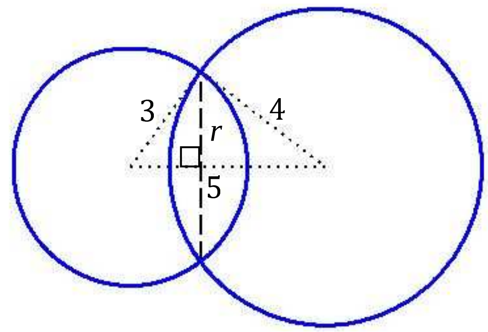
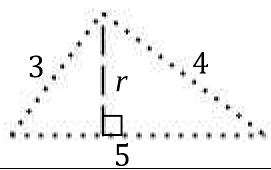
   

   Answer: 9

6. $x|x|=\begin{cases}-x^2, & x<0\\ x^2, & x\ge0\end{cases}$. Thus, the graph of the left hand side of the equation, $x|x|$, is symmetric about the origin. The right hand side of the equation, $2012x+1$, is a line with a positive slope. Hence, their graphs resemble the plot shown, which shows 3 points of intersection.

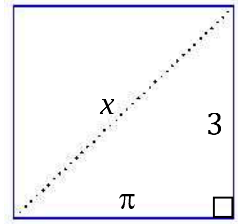
   

   Answer: c

7.	Drawing the height gives a 30-60-90 right triangle (on the right), giving a height of 2 and adjacent side of angle α (of the smaller triangle) a length of $\sqrt{9-4}=\sqrt{5}$. Thus, $\cos(\alpha)=\dfrac{\sqrt{5}}{3}$. Answer: c 
8.	Arranged in numerical order, the numbers are: 12, x, y, 24, 27, 27, 30, where x and y are the only unknown values. Since 27 is *the* mode, we know $x \neq y$, so we take $x<y$. The average is given by $\mu=\dfrac{12+x+y+24+27+27+30}{7}=\dfrac{120+x+y}{7}=\dfrac{119+1+x+y}{7}=\dfrac{119}{7}+\dfrac{1+x+y}{7}=17+\dfrac{1+x+y}{7}$. The average being an integer tells us that $\dfrac{1+x+y}{7}$ must an integer, thus $1+x+y$ must be a multiple of 7. The smallest possible values for x and y are 13 and 14, which would give $\mu=21$. Thus, the mean cannot be less than 21. The largest values for x and y are 22 and 23 which would give $23<\mu<24$. This tells us that all other integer values, from 21 through 23, for μ are obtainable. Answer: a
9.	Taking a cross-section through the centers of both spheres and drawing each radii so they meet at the top point of intersection gives the diagram shown (the dashed line is the diameter of the circle of intersection). The two smaller triangles formed (shown below) are both similar to the larger 3-4-5 right triangle. Hence, we get the relation: $\dfrac{r}{4}=\dfrac{3}{5}\Rightarrow r=\dfrac{12}{5}$.   Answer: $\dfrac{12}{5}=2.4$
10.	A sum of 7 requires either: 1 and 6, or 2 and 5, or 3 and 4. The probability of rolling a 1 is $\dfrac{2}{12}$ and $\dfrac{2}{11}$ for rolling a 6. Thus, the probability of rolling a 1 and 6 or 6 and 1 is $\dfrac{2}{12}\cdot\dfrac{2}{11}\cdot 2=\dfrac{2}{33}$. The probability of rolling a 2 and 5 or 5 and 2 is the same, as is for a 3 and 4 or 4 and 3. Therefore, the probability of rolling a sum of 7 is $\dfrac{2}{33}+\dfrac{2}{33}+\dfrac{2}{33}=\dfrac{2}{11}$. Answer: c
11.	Since **OBAMA** is a 5-digit number and **ROMNEY** is a 6-digit number, **R** = 1. **Y** = 2 implies that **A** must be a 1 or 6, but **R** = 1 so that **A** = 6. The **O**'s in **OBAMA** have another **O** as the sum in the corresponding column. Since **O** cannot be a zero (if it were, we could not get a 6-digit sum), it must be a 9 and there must be a 1 that carries from the next column (so that 9+9+1 gives a 9 in the column containing all **O**'s). **E** must be an odd digit, since **M**+**M**=2**M** is even, then add one from the next column. By similar reasoning, **M** must also be odd (**B**+**B**+1). **M** cannot be 5, because then **E** would be 1, which is already represented by **R**. So, **M** is either 3 or 7, but **M+M+1** must have a 1 that carries to the previous column -- otherwise **N** would have to be a 2, which is already represented by **Y**. Therefore , **M** = 7 and **E**=5, leaving only the digits 4 and 8 for **B**. **B** cannot be 4, since we need a 1 to carry to the previous column. Hence, **B** = 8 giving **OBAMA** = 98676 (and **ROMNEY** = 197352). Answer: 98676

12. The shortest distance, *x*, is the straight line from *Start* to *End* along the curved cylindrical wall. Cutting the cylinder vertically through the *Start* and *End* points and laying it flat gives the rectangle shown. The Pythagorean Theorem gives $x^2 = 3^2 + \pi^2 \Rightarrow x = \sqrt{9+\pi^2}$.

Answer: $\sqrt{9+\pi^2}$



13. $\cos^4(x) - \sin^4(x) = \frac{1}{2} \Rightarrow \underbrace{\left[\cos^2(x)+\sin^2(x)\right]}_{1}\left[\cos^2(x)-\sin^2(x)\right] = \frac{1}{2}$

$\Rightarrow \cos^2(x) - \sin^2(x) = \frac{1}{2} \Rightarrow \cos^2(x) - \left[1-\cos^2(x)\right] = \frac{1}{2} \Rightarrow 2\cos^2(x) - 1 = \frac{1}{2} \Rightarrow 4\cos^2(x) = 3$

$\Rightarrow \cos(x) = \pm\frac{\sqrt{3}}{2} \Rightarrow x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6} \Rightarrow \tan(x) = \pm\frac{1}{\sqrt{3}} \Rightarrow \tan^4(x) = \frac{1}{9}$

Answer: a

14. $\frac{1}{x} + \frac{1}{y} = \frac{1}{5} \Rightarrow \frac{x+y}{xy} = \frac{1}{5} \Rightarrow xy = 5(x+y)$ Thus, we seek a fraction whose numerator is five times the sum of factors of the denominator.

$\frac{1}{5} = \frac{2}{10} = \frac{3}{15} = \frac{4}{20} = \frac{5}{25} = \frac{6}{30} \Rightarrow \frac{6}{30} = \frac{1+5}{30} = \frac{1}{30} + \frac{5}{30} = \frac{1}{30} + \frac{1}{6} \Rightarrow x + y = 36.$

Answer: 36

15. Let the number of pennies, nickels, and dimes be represented by *p*, *n*, and *d*. Thus, $p + 5n + 10d = 100$ and $p + n + d = 45$. $p + 5n + 10d = 100 \Rightarrow p = 100 - 5n - 10d$ substituting into the other equation gives: $100 - 5n - 10d + n + d = 45 \Rightarrow n = \frac{55-9d}{4}$. Since *n* must be non-negative, d can be only 0, 1, 2, 3, 4, 5, or 6. Only $d = 3$ gives an integer value for *n*, 7.

Answer: 7

16. $\log_2(x) + \log_x(2) = 4 \Rightarrow \frac{\log_2(x)}{\log_2(2)} + \frac{\log_2(2)}{\log_2(x)} = 4 \Rightarrow \log_2(x)\left(\frac{\log_2(x)}{1} + \frac{1}{\log_2(x)}\right) = 4\log_2(x)$

$\Rightarrow \left[\log_2(x)\right]^2 + 1 = 4\log_2(x) \Rightarrow \left[\log_2(x)\right]^2 - 4\log_2(x) + 1 = 0$ Now solve for $\log_2(x)$ using the quadratic formula: $\log_2(x) = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} \Rightarrow x = 2^{2-\sqrt{3}}$ or $2^{2+\sqrt{3}}$

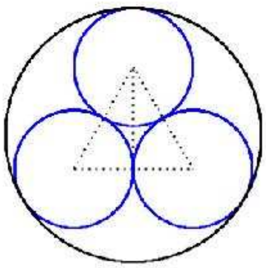
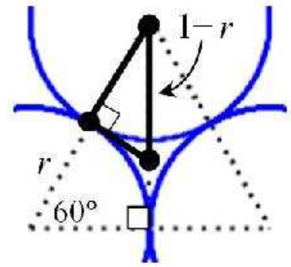
The product of the two solutions is $2^{2-\sqrt{3}} \cdot 2^{2+\sqrt{3}} = 2^4 = 16.$

Answer: 16

17. If $x = 2 - i$ is a root of $f(x)$, then so is its complex conjugate $x = 2 + i$. Thus, $x = 2 \pm i \Rightarrow x - 2 = \pm i \Rightarrow (x-2)^2 = (\pm i)^2 \Rightarrow x^2 - 4x + 4 = -1 \Rightarrow x^2 - 4x + 5 = 0$, which tells us the polynomial $x^2 - 4x + 5$ is a factor of $f(x)$. Dividing $f(x)$ by $x^2 - 4x + 5$ gives another factor of *f*. Performing long division, gives $\frac{2x^4 - x^3 - 22x^2 + 51x - 20}{x^2 - 4x + 5} = 2x^2 + 7x - 4$, which factors as $(2x-1)(x+4)$. Thus, $f(x) = 2x^4 - x^3 - 22x^2 + 51x - 20 = (2x-1)(x+4)\left(x^2 - 4x + 5\right)$, which gives the other two (real) roots $(2x-1)(x+4) = 0 \Rightarrow x = \frac{1}{2}, -4 \Rightarrow \text{sum} = -\frac{7}{2}$.

Answer: $-\frac{7}{2} = -3.5$

Also, the sum of the roots is the negative of the coefficient of the x^3 term, after the polynomial has been "adjusted" so it has a leading coefficient of 1. Dividing $2x^4 - x^3 - 22x^2 + 51x - 20 = 0$ by 2

	gives $x^4-\frac{1}{2}x^3-11x^2+\frac{51}{2}x-10=0$, thus the sum of all the roots is $\frac{1}{2}$. Letting R represent the sum of the (two) real roots, we get $R+(2-i)+(2+i)=\frac{1}{2}\Rightarrow R=-\frac{7}{2}$.
18.	 Analyzing the diagram on the right gives: $\frac{r}{1-r}=\frac{\sqrt{3}}{2}$, since the highlighted triangle is a 30-60-90 right triangle. Thus, $2r=\sqrt{3}-\sqrt{3}r$ $\Rightarrow 2r+\sqrt{3}r=\sqrt{3}\Rightarrow r=\frac{\sqrt{3}}{2+\sqrt{3}}$ Now rationalize the denominator: $r=\frac{\sqrt{3}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2\sqrt{3}-3}{4-3}$ $\Rightarrow r=2\sqrt{3}-3$ **Answer: d** 
19.	There are 5! = 120 ways for the 5 of them to be seated without any restrictions. Now let's count the number of seating arrangements if Bob and Erin *do* sit together. Treating them as one person, gives 4! =24 ways to be seated, but Bob and Erin can switch seats with each other. So, there are 2·4! = 48 ways for them to be seated with Bob and Erin next to each other. Thus, there are 120 – 48 = 72 ways for the 5 of them to be seated so that Bob and Erin are *not* next to each other. **Answer: c**
20.	If the first logician did not want coffee, then he/she would have been able to answer "No." Thus, he/she must want coffee and does not know if the rest of his party does. The same applies to the next two logicians. Now that the fourth logician knows that the other three want coffee, he/she can give a definitive answer. He/she says "No" which indicates that he/she does not want coffee. Thus, only 3 of them want coffee. **Answer: c**

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