New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2013

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. The function $f(x)$ satisfies the equation $f(x) = f(x+1) + f(x-1)$ for all values of x. If $f(1) = 1$ and $f(2) = 2$, what is the value of $f(2013)$?

2. The number 20 has a total of 6 factors (i.e. 1, 2, 4, 5, 10, and 20). 1,000,000,000 has a total of how many factors?

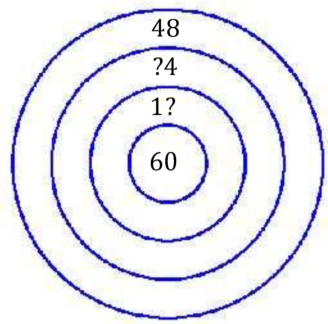
3. Arranged in ascending order (i.e. from smallest to largest): $2^{555}, 3^{333}, 5^{222}$, we obtain
   a) $2^{555}, 3^{333}, 5^{222}$ b) $2^{555}, 5^{222}, 3^{333}$ c) $3^{333}, 5^{222}, 2^{555}$ d) $5^{222}, 3^{333}, 2^{555}$

4. In a camp, 80 men have food for 100 days. After 5 days, 10 men leave. After 10 more days, 20 men leave. How many more days of food is there for the remaining men?

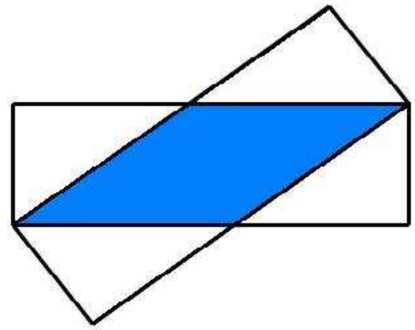
5. Suppose 10 distinct lines are drawn on a page. What is the *maximum* number of intersection points that can be obtained?

6. The equation $\sqrt{2x+2\sqrt{x}} + \dfrac{2}{\sqrt{2x+2\sqrt{x}}} = 3$, has two real solutions. What is their sum?

7. The diagram shows a dartboard with points awarded for hitting each region, but two digits are missing. I threw three darts, hit three different regions and scored 132. My friend also threw three darts, hit three different regions and scored only 90. What is the sum of the missing digits?
   a) 9 b) 10 c) 12 d) 13



8. Two rectangles that measure 1 by 3 intersect so that two opposite corners coincide, as shown. What is the area of the shaded region (i.e. the overlap)?



a) $\frac{3}{2}$ b) $\frac{\sqrt{10}}{2}$ c) $\frac{5}{3}$ d) NOTA

9. The factorial of a positive integer is defined as: $n! = n(n-1)(n-2)\cdots 1$, for example $3! = 3\cdot 2\cdot 1 = 6$. Which of the following statements is/are true?

I. $n! \leq n^n$, for *all* positive integers n.
II. $n!$ cannot be a perfect square for *all* integers $n \geq 2$.
III. *Every* positive rational number can be expressed as the ratio of factorials or the ratio of products of factorials. For example: $\frac{1}{20} = \frac{3!}{5!}$, and $\frac{7}{8} = \frac{7!}{2!4!5!}$.

a) I only b) I and II only c) II and III only d) I, II, and III

10. How many positive integer solutions are there to the equation $2x + 3y = 2013$? For example: $(0, 671)$ is *one* solution, i.e. $x = 0$ and $y = 671$, but is 0 is not positive, so this solution is not among the ones we seek.

a) 335 b) 336 c) 402 d) 403

11. Two cars start together around a three mile race track, both travelling in the clockwise direction. Car A travels at 60 miles/hr, while Car B travels at 80 miles/hr. How many minutes, after the start, does it take Car B to *catch up* to Car A?

12. What value of x solves the equation $\log_3(x) + \log_9(x) + \log_{\frac{1}{9}}(x) = 2013$?

Note: The bases of the logarithms are 3, 9, and $\frac{1}{9}$, respectively.

13. If $\tan(x) + \sec(x) = 2$, then what is the numerical value of $\sin(x)$?

14. The graphs of $x^2 + xy + x = 1$ and $y^2 + xy + y = 11$ intersect at exactly two points: (x_1, y_1) and (x_2, y_2). What is the sum of all the coordinates (i.e. $x_1 + y_1 + x_2 + y_2$)?

15. A store's policy is that you can exchange 4 empty soda bottles for 1 new bottle of soda. I bought 50 bottles for a party. At most, how many more bottles of soda can I get by exchanging?

a) 12 b) 16 c) 17 d) 24

16. Given that $\log_{10}(2) \approx 0.301$, how many digits are in the expansion of 5^{100}?

17. The greatest integer function, denoted $[\![x]\!]$, is defined as *the greatest integer less than or equal to x*. For example, $[\![1.9]\!] = 1$ and $[\![-1.9]\!] = -2$. What is the *smallest* real number, *x*, that solves the equation $[\![x]\!] + [\![2x]\!] + [\![3x]\!] = 2013$?

18. What is the remainder when the polynomial $x^{2013} + x + 1$ is divided by $x^2 - 1$?
a) $x - 2013$ b) $x + 2013$ c) $2x - 1$ d) $2x + 1$

19. A monkey has been trained to type all the letters of the alphabet exactly once, but the order is entirely random. What is the probability the word "math" appears somewhere in the string of 26 letters? (See problem #9 for the definition of factorial, "$!$".)
a) $\frac{4!}{26!}$ b) $\frac{23!}{26!}$ c) $\frac{4!\,22!}{26!}$ d) $\frac{4}{26}$

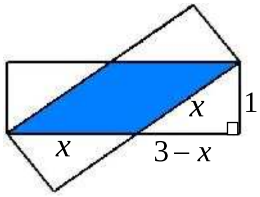
20. Suppose we have 5 bags that each contain 10 gold coins. One bag, which remains to be identified, contains all counterfeit coins. All the coins appear identical (i.e. by look and feel). However, the genuine coins each weigh 10 grams, while the counterfeit coins are slightly heavier, weighing 10.1 grams. What is the *fewest* number of weighings, using a standard digital scale, needed to <u>guarantee</u> the bag of counterfeit coins is identified? Note: We can open each bag and remove as many coins as we would like for the weighings.
a) 1 b) 2 c) 3 d) 4

NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2013 ~ Solutions

1.	Solving for $f(x+1)$ gives ① $f(x+1)=f(x)-f(x-1)$ which also gives $f(x)=f(x-1)-f(x-2)$ and $f(x-1)=f(x-2)-f(x-3)$. Substituting these last two expressions into ① gives $f(x+1)=\left[f(x-1)-f(x-2)\right]-\left[f(x-2)-f(x-3)\right]$, ② $f(x+1)=f(x-1)-2f(x-2)+f(x-3)$. Now replace $f(x-1)$ with $f(x-2)-f(x-3)$ in ② to obtain $f(x+1)=-f(x-2)$. Thus, $f(2013)=-f(2010)=f(2007)=-f(2004)=\ldots=(-1)^n f(2013-3n)$. Now let $n=671$ to obtain $f(2013)=(-1)^{671}f(0)=-f(0)$, but $f(0)$ satisfies the equation $f(1)=f(0)+f(2)$. Therefore, $f(0)=-1$ giving $f(2013)=-(-1)=1$. **Answer: $f(2013)=1$**
2.	$1{,}000{,}000{,}000=10^9=(2\cdot 5)^9=2^9\cdot 5^9$. Hence, factors of 10^9 are numbers of the form $2^m\cdot 5^n$, where $m,n\in\{0,1,2,3,4,5,6,7,8,9\}$. Thus, each prime factor may have 10 possible exponents for a total of $10\cdot 10=100$ different factors. **Answer: 100**
3.	$2^{555}=2^{5\cdot 111}=\left(2^5\right)^{111}=32^{111}$, $3^{333}=3^{3\cdot 111}=\left(3^3\right)^{111}=27^{111}$, and $5^{222}=5^{2\cdot 111}=\left(5^2\right)^{111}=25^{111}$. Since $25^{111}<27^{111}<32^{111}$, $5^{222}<3^{333}<2^{555}$. **Answer: d**
4.	For convenience, let the amount of food that each man requires per day be 1 unit. Thus, 100 days of food for 80 men requires $80\cdot 100=8000$ units. After 5 days, $5\cdot 80=400$ units of food have been consumed, leaving 7600 units for the remaining 70 men. After 10 more days, $10\cdot 70=700$ additional units have been consumed, leaving 6900 units for the remaining 50 men. Thus, there are $6900\div 50=138$ days of food for the 50 men. **Answer: 138**
5.	For two lines, there is one point of intersection. For three lines, there are three points of intersection. Each time we add a line, it can intersect (at most) all the other lines. Thus, adding the n^{th} line adds (at most) an additional $n-1$ intersection points. Therefore, by the 10^{th} line, we have at (at most) $1+2+3+4+5+6+7+8+9=45$ points of intersection. **Answer: 45**
6.	Letting $y=\sqrt{2x+2\sqrt{x}}$, gives $y+\dfrac{2}{y}=3 \Rightarrow y^2-3y+2=0 \Rightarrow (y-1)(y-2)=0 \Rightarrow y=1,\ y=2$. $\Rightarrow \sqrt{2x+2\sqrt{x}}=1$, which gives $x=1-\tfrac{1}{2}\sqrt{3}$ or $\sqrt{2x+2\sqrt{x}}=2$, which gives $x=1$. Thus, the sum is $1-\tfrac{1}{2}\sqrt{3}+1=2-\tfrac{1}{2}\sqrt{3}$. **Answer: $2-\tfrac{1}{2}\sqrt{3}$**
7.	Let the missing digits be x and y, so that the unknown point values are $10x+4$ and $10+y$. Listing all possible point totals by hitting three different regions gives: ① $60+(10+y)+(10x+4)=\boxed{74+10x+y}$, ② $60+(10+y)+48=\boxed{118+y}$ ③ $60+(10x+4)+48=\boxed{112+10x}$, and ④ $(10+y)+(10x+4)+48=\boxed{62+10x+y}$ A score of 90 can only be obtained by ① or ④, and a score of 132 cannot be obtained by ②. If ① gives 90, then ④ cannot give 132, since it is 12 points less than ①. Thus, if ① gives 90, then ③ must give 132. Solving $74+10x+y=90$ and $112+10x=132$ gives no solution. Now try ④ giving 90 and ③ giving 132 $\Rightarrow 62+10x+y=90$ and $112+10x=132 \Rightarrow x=2,\ y=8$. **Answer: b**

8.	 Using basic geometry, we see all four right triangles formed are congruent. The Pythagorean Theorem gives $(3-x)^2+1^2=x^2 \Rightarrow 10-6x=0 \Rightarrow x=\frac{5}{3}$. The area of the rhombus is base (x) times height (1), giving $\frac{5}{3}\cdot 1$. Answer: c
9.	Statement I is clearly true, since $n^n=\underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{n} \geq \underbrace{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1}_{n}=n!$, with equality only for $n=1$. Statement III is true, since any positive rational number $\frac{p}{q}$ can be expressed as $\frac{p!(q-1)!}{(p-1)!q}$. Thus, the only choice that includes both I and III, is choice d. Statement II is also true! Answer: d
10.	Solving for y gives $y=\frac{2013-2x}{3}=671-\frac{2}{3}x$, which is an integer only when x is a multiple of 3 and positive only when $x<\frac{2013}{2}=1006.5$. Thus, we need to count the number of multiples of 3 greater than 0 and less than or equal to 1006. They are: 3, 6, 9, … , 1005, which is $\frac{1005}{3}=335$ values. Answer: a
11.	When Car B catches Car A (for the first time), it will have gone one extra lap (of 3 miles). Hence, we simply need to solve: $80t=60t+3$, where t is in hours. $t=0.15$ hr = 9 minutes. Answer: 9
12.	$\log_3(x)+\log_9(x)+\log_{\frac{1}{9}}(x)=2013$ can be written as $\log_3(x)+\frac{\log_3(x)}{\log_3(9)}+\frac{\log_3(x)}{\log_3\left(\frac{1}{9}\right)}=2013$, which simplifies to $\log_3(x)+\frac{\log_3(x)}{2}+\frac{\log_3(x)}{-2}=2013$, which further simplifies to $\log_3(x)=2013$. This gives $x=3^{2013}$. Answer: 3^{2013}
13.	$\tan(x)+\sec(x)=2 \Rightarrow \frac{\sin(x)}{\cos(x)}+\frac{1}{\cos(x)}=2 \Rightarrow \frac{\sin(x)+1}{\cos(x)}=2 \Rightarrow \sin(x)+1=2\cos(x)$, now square both sides (so we can use the Pythagorean Identity to express $\cos^2(x)$ in terms of $\sin^2(x)$). Thus, $\sin^2(x)+2\sin(x)+1=4\cos^2(x) \Rightarrow \sin^2(x)+2\sin(x)+1=4\left(1-\sin^2(x)\right) \Rightarrow 5\sin^2(x)+2\sin(x)-3=0$ Which can be solved by factoring: $\left(5\sin(x)-3\right)\left(\sin(x)+1\right)=0 \Rightarrow \sin(x)=\frac{3}{5}$ or $\sin(x)=-1$, but if $\sin(x)=-1$, $\cos(x)=0$ (and $\tan(x)$ is undefined). Hence, $\sin(x)=\frac{3}{5}$. Answer: $\frac{3}{5}$
14.	Combining the two equations by adding gives: $\left(x^2+xy+x\right)+\left(y^2+xy+y\right)=1+11 \Rightarrow$ $\left(x^2+2xy+y^2\right)+(x+y)=12 \Rightarrow (x+y)^2+(x+y)-12=0 \Rightarrow \left((x+y)+4\right)\left((x+y)-3\right)=0$ $x+y=-4$ or $x+y=3 \Rightarrow$ sum of all coordinates is -1 Answer: -1
15.	Initially, 48 of the 50 bottles can be exchanged, for 12 new bottles of soda, with 2 bottles not exchanged. Then I will have 14 bottles (12+2) for exchange, but can only exchange 12 of them, for 3 new bottles, with 2 not exchanged. Then, I will have 5 bottles (3+2) for exchange, but can only exchange 4 of them for 1 new bottle, with 1 not exchanged. Then I will have 2 bottles and can no longer return for exchange. Thus, the maximum number of new bottles is 12+3+1 = 16. Answer: b

16.	Let $x = 5^{100}$, multiply by 2^{100}: $2^{100}x = 2^{100} \cdot 5^{100} \Rightarrow 2^{100}x = 10^{100}$ Now take log base 10 of both sides: $\log\left(2^{100}x\right) = 100 \Rightarrow \log\left(2^{100}\right) + \log(x) = 100 \Rightarrow 100\log(2) + \log(x) = 100 \Rightarrow$ $\log(x) = 100 - 100\log(2) \approx 100 - 100 \cdot 0.301 = 69.9 \Rightarrow x \approx 10^{69.9}$ which has 70 digits **Answer: 70**
17.	To get an approximation solve $x + 2x + 3x = 2013$, which gives $x = \frac{2013}{6} = 335.5$. Substituting this into the equation: $[\![335.5]\!] + [\![2 \cdot 335.5]\!] + [\![3 \cdot 335.5]\!] = [\![335.5]\!] + [\![671]\!] + [\![1006.5]\!] = 335 + 671 + 1006 = 2012$, which is too small (by 1). Thus, we need to increase the 335.5 just enough so that only one of the terms gets incremented by 1. The last term, i.e. $[\![3x]\!]$, is the one we focus on since the 3 multiplier makes the expression increase more for smaller changes in x. We need $3x = 1007$. Thus, $x = \frac{1007}{3} = 335\frac{2}{3}$. **Answer: $\frac{1007}{3} = 335\frac{2}{3}$**
18.	$\left(x^{2013} + x + 1\right) \div \left(x^2 - 1\right) = Q(x) + \dfrac{R(x)}{x^2 - 1}$, where $Q(x)$ is the quotient and $R(x)$ is the remainder. Equivalently, $x^{2013} + x + 1 = Q(x) \cdot \left(x^2 - 1\right) + R(x)$. Since the divisor is a 2nd degree polynomial, the remainder must be a 1st degree polynomial (or smaller). Thus, $R(x)$ is of the form $ax + b$. Hence, we can write $x^{2013} + x + 1 = Q(x) \cdot \left(x^2 - 1\right) + ax + b$. Letting $x = -1$ and $x = 1$ will give two equations in terms of a and b. $x = -1$ gives $(-1)^{2013} + (-1) + 1 = Q(x) \cdot 0 + a(-1) + b \Rightarrow$ ① $-1 = -a + b$, and $x = 1$ gives $(1)^{2013} + 1 + 1 = Q(x) \cdot 0 + a(1) + b \Rightarrow$ ② $3 = a + b$. Solving ① and ②, gives $a = 2$ and $b = 1$. **Answer: d**
19.	The word "math" may appear in any one of 23 positions within the 26 letter string (i.e. math***...*, *math***...*, **math***...*, ... , ***...**math*, ***...*math), while the remaining 22 letters can be arranged in any order for 22! ways. For a total of $23 \cdot 22! = 23!$ ways to have the word "math" within the string. There are a total of 26! ways to arrange the 26 letters of the alphabet. Thus, the probability that the word "math" appears is $\frac{23!}{26!}$. **Answer: b**
20.	The bag of counterfeit coins can be identified with only 1 weighing! Number each bag 1, 2, 3, 4, and 5. From the bag labeled 1 take 1 coin, from the bag labeled 2 take 2 coins, and so on until 5 coins are removed from the bag labeled 5. Now weigh them all at once. If there were no counterfeits, the 15 coins (1+2+3+4+5) would weigh 150 grams. But, because of the counterfeits, the total weight will be heavier. If the total weight is 150.3 grams, then we know there are 3 counterfeit coins (each being 0.1 gram heavier than the genuine coins), which must have come from bag number 3. If the total weight is 150.1 grams, then we know there is only 1 counterfeit coin and it came from bag number 1. Similarly for 150.2 grams, 150.4 grams, and 150.5 grams. **Answer: a**

NYSMATYC 1967