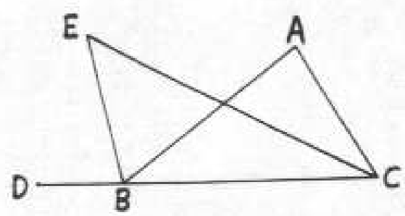
SPRING 2001

1. If this triangle was continued, what number would be directly below 144?

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16
•
•
•

2. How many digits does the number $25^{501} \bullet 6^{3} \bullet 2^{1004}$ have?

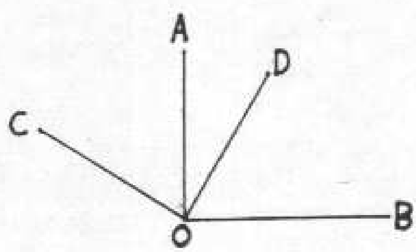
3. $\overrightarrow{CE}$ bisects $\angle ACB$, $\overrightarrow{BE}$ bisects $\angle ABD$, and the measure of angle A is $80°$ Find the measure of angle E.



4. In an El Niño year, the probability of spring drought in Cantonville is 60%. In a normal year it is 10%. In the next century one-fifth of the years are expected to be El Niño years. Find the expected number of drought years in Cantonville in the next century.

A) 10 B) 20 C) 35 D) 40

5. If the measure of angles AOB and COD are both $90°$, find the measure of COB added to the measure of AOD.



6. Let $(m+1)^2 x - my - m^2 - 1 = 0$ be the equation of a family of lines where m is any real number. Through what fixed point do all lines in this family pass?

7. How many positive integral solution pairs are there to the equation $n^2 - m^2 = 270$?

A) 0 B) 1 C) 2 D) Infinitely many

8. A two digit number is multiplied by 23, which yields a four digit number that has a 1 as its first and last digit and the original two digits as its second and third digits $(xy \bullet 23 = 1xy1)$. What is the original number?

9. The first 27 numbers in the sequence 3, 33, 333, 3333, 33333, . . . , are added together. What is the digit in the hundreds place in their sum?

A) 1 B) 3 C) 5 D) 6

10. A train traveling at 100 km/hr. takes 3 seconds to enter a tunnel and an additional 30 seconds to pass completely through it. What is the length of the train?

11. In a certain football game a team can score only 3 or 7 points at any one time. Is there a largest integer that could not be a team score in this game? If so, what is it?

12. Albert Einstein was once asked how many students he had. He replied, "One-half of them study only mathematics, $\frac{1}{3}$ of them study only geometry, $\frac{1}{7}$ study only chemistry, and there are 20 who don't study at all." Assuming geometry is included in Mathematics and these are the only students, then how many students did he have?

13. We know a young lad in Dundee
Whose age had its last digit three.
The square of the first
is his whole age reversed.
So what must this laddy's age be?

14. A ball was floating in a lake when the lake froze. The ball was removed without breaking the ice, leaving a hole 24 cm across and 8 cm deep. What was the radius of the ball in centimeters?

A) 8 B) 12 C) 13 D) $8\sqrt{13}$

15. For $x \geq 0$, define f(x) = the largest integer less than $\sqrt[3]{x}$ minus the smallest integer greater than or equal to $\sqrt{x}$.
What is the value of f(47) + f(64)?

16. Find the sum of the infinite series

$$\frac{1}{2}+\frac{2^3-3}{2^2\bullet 3}+\frac{2^5-3^2}{2^3\bullet 3^2}+\frac{2^7-3^3}{2^4\bullet 3^3}+\frac{2^9-3^4}{2^5\bullet 3^4}$$

A) $\frac{3}{2}$ B) 2 C) 3 D) $4\frac{1}{6}$

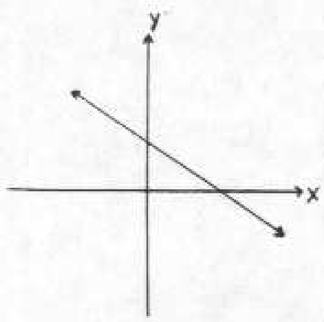
17. The mean of three numbers is m. If the first number is doubled, the second tripled, and the third is unchanged, the mean remains the same. On the other hand, if you double the second, triple the third and leave the first unchanged, the mean will be quadrupled. The sum of the squares of the original three numbers is

A) $8m^2$ B) $14m^2$ C) $21m^2$ D) $27m^2$

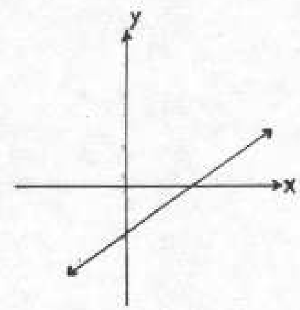
18. If the graphs of $y = x^2 + bx + c$, $y = \frac{2}{3}x - 4$, and $y = -x^2 + c$ have a common intersection point on the x-axis, then find the value of b.

19. Which of the following is the best graph of $\frac{3^{x+y}}{2^{x-y}} = 6$?

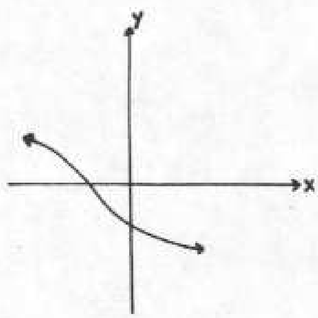
A)



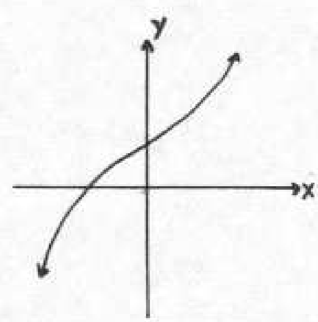
B)



C)



D)



20. If θ is an obtuse angle in degrees, then exactly how many of the following could not be equal to $\tan^{2001}\left(\frac{\theta + 90°}{2}\right)$

i) $\frac{1-\pi^e}{\sqrt{2}}$ ii) $\left(\frac{8^{7000} - 9^{15000}}{6^{13000} - 7^{11000}}\right)^{2/3}$ iii) $-e\pi\sqrt{29}$ iv) $-\frac{\sqrt[3]{7}}{12}$?

A) None B) 1 C) 2 D) All four

NYSMATYC MATH LEAGUE COMPETITION
Spring 2001

ANSWER SHEET

Directions: You have one full hour to take this test. Scrap paper is allowed. The use of a calculator is permitted, but not stored programs on the calculator. Moreover, books, tables, and computers are not permitted. You are not expected to answer all problems. Yet, don't waste too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer and nothing is deducted for a blank. No partial credit.

Name ANSWER KEY **College** ______
Home Address ______
(include city, state, zip code)
Name of teacher in whose class you are enrolled ______

1.	168	11.	11
2.	1005	12.	56
3.	40°	13.	63
4.	B (20 yrs.)	14.	13 cm
5.	180°	15.	-9
6.	(1, 2)	16.	B (2)
7.	A (0)	17.	C (21 m^2)
8.	77	18.	-12
9.	B (3)	19.	A
10.	$\frac{1}{12}$ km	20.	C (2)

Correct ______ X 4 = ______

Incorrect ______ X –1 = ______

Total ______

SOLUTIONS
SPRING 2001

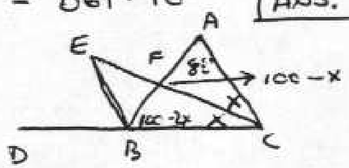
1) THE NUMBER ON THE RIGHT END IS ALWAYS A PERFECT SQUARE. IT IS ALWAYS RIGHT ABOVE THE NEXT TO THE LAST NUMBER IN THE LINE BELOW. SINCE $144 = 12^2$, THE NEXT LINE ENDS IN $13^2 = 169$. SO 144 IS ABOVE 168 OR 168 IS DIRECTLY BELOW 144. **ANS 168**

2) $25^{501} \cdot 6^3 \cdot 2^{1004} = (5^2)^{501} \cdot 6^3 \cdot 2^{1004}$

$= 5^{1002} \cdot 6^3 \cdot 2^{1002} \cdot 2^2 = (5 \cdot 2)^{1002} \cdot 6^3 \cdot 2^2$

$= 864 \cdot 10^{1002}$ **ANS. 1005**

3)



$\angle AFC + \angle A + \angle ACF = 180$

$\angle AFC + 80 + x = 180$

$\angle AFC = 100 - x$

$\therefore \angle AFC = 100 - x = \angle EFB$

$\angle ABC + \angle A + \angle C = 180$

$\angle ABC + 80 + 2x = 180$

$\angle ABC = 100 - 2x$

$\angle DBA + \angle ABC = 180$ } $\angle DBA = 80 + 2x$

$\angle DBA + 100 - 2x = 180$ } $\angle EBA = 40 + x$

$\angle E + \angle EBA + \angle EFB = 180$

$\angle E + 40 + x + 100 - x = 180$

$\angle E = 40°$ ANS.

NOTE: FOR THESE CONDITIONS $\angle E = \frac{1}{2} \angle A$.

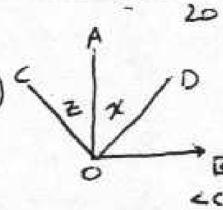
4) $\frac{1}{5}(100) = 20$ EL NIÑO YRS. 80 NORMAL

$20(.60) = 12$ DROUGHT YRS. IN EL NIÑO

$80(.10) = 8$ DROUGHT YRS. IN NORMAL

20 " " **ANS. B**

5)



NOTE: $x + z = 90°$ GIVEN

$\angle COB = 90 + z$

$\angle AOD = x$

$\angle COB + \angle AOD = 90 + x + z = 90 + 90$

$= 180°$ ANS.

6) FOR $m = 0$ WE GET $x - 1 = 0 \Rightarrow x = 1$ A VERTICAL LINE.
FOR $m = -1$ WE GET $y - 1 - 1 = 0 \Rightarrow y = 2$ A HORIZONTAL LINE.
THE POINT $(1,2)$ IS ON BOTH.
HENCE, **$(1,2)$ ANS.**

7) $n^2 - m^2 = (n+m)(n-m) = 270$
IN PRIME FACTORS $270 = 2 \cdot 3^3 \cdot 5$.
NOW $n+m$ AND $n-m$ ARE BOTH EVEN OR BOTH ODD. HOWEVER, 270 HAS ONLY ONE FACTOR OF 2 SO ITS FACTORS ARE ONE EVEN AND ONE ODD. SO NO INTEGRAL FACTORS. **ANS. 0** **A**

8) $xy \cdot 23 = 1xy1$, THUS, $y = 7$ AS $7 \cdot 3 = 21$

$x7 \cdot 23 = 1x71$ WRITING IN EXPANDED FORM GIVES $(10x+7)23 = 1000 + 100x + 10(7) + 1$

OR $230x + 161 = 1071 + 100x$

$130x = 910 \Rightarrow x = 7$ **ANS. 77**

9) THE ONE'S PLACE HAS 27 THREES THAT ADD TO 81. THE 10'S PLACE HAS 26 THREE'S THAT ADD TO 78 CARRYING 8 THE 10'S DIGIT IN THE SUM IS 6 AND WE CARRY 8. THERE ARE 25 THREE'S IN THE HUNDRED'S PLACE THAT ADD TO 75. $75 + 8 = 83$
3 IS IN HUNDREDS PLACE. **ANS. B**

10) SINCE IT TAKES 3 SECONDS TO ENTER THE TUNNEL, THE LENGTH OF THE TRAIN IS
$(\text{RATE})(\text{TIME}) = \frac{1}{36}\frac{\text{km}}{\text{sec}} \cdot 3\,\text{sec} \cdot \frac{1\,\text{km}}{36\,\text{sec}} \cdot \frac{1\,\text{km}}{12}$

$= \frac{1}{12}$ KM

11) ANY POSITIVE INTEGER CAN BE WRITTEN AS $3n$, $3n+1$, OR $3n+2$. 7 IS OF THE FORM $3n+1$ WHILE 14 IS OF THE FORM $3n+2$. THUS, ANY INTEGER 14 OR GREATER CAN BE WRITTEN IN TERMS OF 3'S AND 7'S.
$13 = 7 + 6 = 7 + 2 \cdot 3$ $12 = 4 \cdot 3$ $11 = 9 + 2 = 4 + 7$
11 CANNOT BE WRITTEN IN TERMS OF 7'S AND 3'S. **ANS. 11**

12) LET x = NUMBER OF STUDENTS.

$\frac{1}{2}x + \frac{1}{7}x + 20 = x$

$7x + 2x + 280 = 14x \Rightarrow 280 = 5x$

$x = 56$ ANS.

13) THE AGE IS OF THE FORM $x3$, WHERE x IS THE 10'S DIGIT, AND PRESUMING THE LAD IS NOT IN THE HUNDREDS. THE PROBLEM BECOMES $x^2 = 30 + x$

OR $x^2 - x - 30 = 0$

$(x-6)(x+5) = 0$

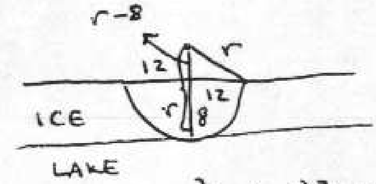
$x = 6$ OR $x = -5$

ANS, $x = 6$ AND

AGE IS **63 ANS**

SOLUTIONS
(CONTINUED)

14)



FROM SKETCH $r^2 = (r-8)^2 + 12^2$

$r^2 = r^2 - 16r + 64 + 144$

$0 = -16r + 208$

$16r = 208 \Rightarrow$ **r = 13 cm** ANS.

15) $\sqrt[3]{47} \doteq 3.$ $\sqrt[3]{64} = 4$

$\sqrt{47} \doteq 6.$ $\sqrt{64} = 8$

$f(47) = 3 - 7 = -4$

$f(64) = 3 - 8 = -5$

$\therefore f(47) + f(64) =$ **−9 ANS.**

16) BY REDUCING AND REWRITING WE GET

$\frac{1}{2} + \left(\frac{2}{3} - \frac{1}{4}\right) + \left(\frac{4}{9} - \frac{1}{8}\right) + \left(\frac{8}{27} - \frac{1}{16}\right) + \left(\frac{16}{81} - \frac{1}{32}\right) + \ldots$

$= \frac{1}{2} + \left(\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right)$

$= \frac{1}{2} + \frac{2}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots\right) - \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$

$= \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{1 - \frac{2}{3}} - \frac{1}{4} \frac{1}{1 - \frac{1}{2}}$

$= \frac{1}{2} + 2 - \frac{1}{2} = 2$ **ANS. B**

17) LET X, Y, AND Z BE THE 3 NOS. THEN

$\frac{x+y+z}{3} = m \Rightarrow$ ① $x + y + z = 3m$

$\frac{2x+3y+z}{3} = m \Rightarrow$ ② $2x + 3y + z = 3m$

$\frac{x+2y+3z}{3} = 4m \Rightarrow$ ③ $x + 2y + 3z = 12m$

DOUBLING ① AND SUBTRACTING FROM ② GIVES

$y - z = -3m$

① − ③ GIVE

$-(y + 2z = 9m)$

$-3z = -12m$

$z = 4m$

$y - z = -3m$

$y - 4m = -3m$

$y = m$

$x + y + z = 3m$

$x + m + 4m = 3m$

$x = -2m$

$x^2 + y^2 + z^2$

$(-2m)^2 + m^2 + (4m)^2$

$= 21m^2$

ANS. C

18) SINCE THE COMMON POINT IS ON THE X-AXIS, $y = 0$.

IN $y = \frac{2}{3}x - 4$, $0 = \frac{2}{3}x - 4 \Rightarrow x = 6$

IN $y = -x^2 + c$, $(6,0)$ GIVES $0 = -6^2 + c$

$c = 36$

IN $y = x^2 + bx + c$ WE GET

$0 = 6^2 + b(6) + 36 \Rightarrow 6b = -72$

$b = -12$

ANS b = −12

19) LET $u = x + y$ AND $v = x - y$ SO

$\frac{3^u}{2^v} = 6 \Rightarrow 3^u = 6 \cdot 2^v$

TAKING LN OF BOTH SIDES GIVES

$\ln 3^u = \ln(6 \cdot 2^v) = \ln 6 + \ln 2^v$

$u \ln 3 = \ln 6 + v \ln 2$

or $u = \frac{\ln 6}{\ln 3} + v \frac{\ln 2}{\ln 3}$

THIS IS A STRAIGHT LINE WITH POSITIVE VERTICAL INTERCEPT AND NEGATIVE SLOPE AS LN2<1.

REPLACING u AND v GIVES

$\left(1 + \frac{\ln 2}{\ln 3}\right) y = \frac{\ln 6}{\ln 3} + \left(\frac{\ln 2}{\ln 3} - 1\right) x$

AGAIN A STRAIGHT LINE WITH NEGATIVE SLOPE AND POSITIVE Y INTERCEPT.

ANS. A

20) θ IS OBTUSE, THUS, $90 < \theta < 180$

$\Rightarrow 180 < \theta + 90 < 270$

$\Rightarrow 90 < \frac{\theta + 90}{2} < 135$ ①

SINCE $\frac{\theta+90}{2}$ IS QUAD II, $\tan\left(\frac{\theta+90}{2}\right) < 0$

ALSO $\tan\left(\frac{\theta+90}{2}\right) < -1$ FOR CONDITION ①

THUS $\tan\left(\frac{\theta+90}{2}\right)^{2001} < -1$.

i) < −1 ii) > 0 BECAUSE OF $\frac{2}{2}$ EXP.

iii) < −1 AND −1 < iv < 0

ELIMINATE ii) & iv)

ANS 2 C