Math League Questions
Spring 2005

1) The profits P (in millions of dollars) for a company are modeled by a quadratic function of the form $P = at^2 + bt + c$, where t represents the year. If you were president of the company which of the models would you prefer?

A) a is positive and $t \le \frac{-b}{2a}$

B) a is positive and $t \ge \frac{-b}{2a}$

C) a is negative and $t \le \frac{-b}{2a}$

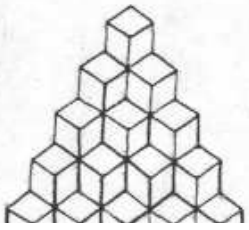
D) a is negative and $t \ge \frac{-b}{2a}$

2) If a family has four children, is it more likely to have two of each sex or three of one sex and one of the other, assuming births of males and females are equally likely?

3) A partition of a natural number is a collection of natural numbers (some of which may be repeated) which add up to the given number. For example, $10 = 2+2+3+3$. Consider natural-number partitions of 13, for example $1+2+2+8=13$, $9+4=13$,... Find a partition of 13 for which the product of all terms is the maximum.

4) Dave computed an answer to be 22.5. However, in the next to last step of the computation he subtracted 7 instead of adding 7. Then in the last step of the computation he multiplied by 0.3 instead of dividing by 0.3. Assuming that Dave computed correctly, what is the correct answer?

5) This tower is made of 35 cubes in 5 layers. How many cubes are needed to form a similar tower with 10 layers?



6) What is the value of
$(\tan(45°))\cdot(\cos(30°))\cdot(\sec(120°))\cdot(\sin(150°))\cdot(\cot(90°))\cdot(\csc(315°))$?

7) If $3 \# 7 = 1$, $4 \# 2 = 2$, and $9 \# 15 = 3$, $10 \# 25 = 5$ what is $6 \# 6$?

8) In the Hope family, there are seven sisters and each sister has one brother. The family has a ten-passenger van. How many family members must stay home when they go to grandmother's house for dinner, assuming each family member must have their own seat?

9) Find the largest four-digit number that has exactly three factors.

10) Let $S_n = 1-2+3-4+\ldots+(-1)^{n-1}n$, where $n = 1,2,3,4,\ldots$ Then $S_{17}+S_{33}+S_{50} =$:
A) 0 B) 1 C) 2 D) -1

11) With 400 members voting the House of Representatives defeated a bill. A revote, with the same members voting, resulted in the passage of the bill by twice the margin by which it was originally defeated. The number voting for the bill on the re-vote was 12/11 of the number voting against it originally. How many more members voted for the bill the second time then voted for it the first time?

A) 75 B) 60 C) 50 D) 45

12) A and B move uniformly along two straight paths intersecting at right angles in point O. When A is at O, B is 500 yards short of O. In 2 minutes they are equidistant from O and in 8 more minutes they are again equidistant from O. Then the ratio of A's speed to B's speed is:

A) 4:5 B) 5:6 C) 2:3 D) 1:2

13) The area of the ring between two concentric circles is 12.5π sq. in. The length of a chord of the larger circle tangent to the smaller circle, in inches, is:

A) $\frac{5}{\sqrt{2}}$ B) 5 C) $5\sqrt{2}$ D) 10

14) The number of points in the same plane equidistant from a circle and two parallel tangents to the circle is:

A) 0 B) 3 C) 2 D) Infinite

15) When written in base b, a number is 743. In base $2b-3$ its 291. What is the number when it's written in base 10?

16) What is the exact product (in exponential or radical form) of the 50 largest of the following numbers?

$$\frac{\sqrt[149]{.999}}{\sqrt[150]{.999}}, \frac{\sqrt[148]{.999}}{\sqrt[149]{.999}}, \frac{\sqrt[147]{.999}}{\sqrt[148]{.999}}, \frac{\sqrt[146]{.999}}{\sqrt[147]{.999}}, \ldots, \frac{\sqrt{.999}}{\sqrt[3]{.999}}$$

17) Find the exact value of $\dfrac{1.5^{-11000}\,0.875^{1000}\,2.25^{4000}}{7^{1002}\,27^{-1000}}$.

18) If $N = 7+8+9+10+\ldots+33+35!+37!+39!+41!+43!+\ldots+2005!$
How many integers between 5 and 20 are NOT factors of N?

19) A data set consists of 6 data items, all integers. The mean = 7, the median = 6.5, the mode = 6, and the smallest data value is 5. Write the data items in size order, smallest first.

20) How many points do the graphs of the two equations intersect?
Given $(x-4)^2 + (y-3)^2 = 5000^{10000}$ and $y = -x^2 - 10,000!$

A) 0 B) 1 C) 2 D) Otherwise

NYSMATYC MATH LEAGUE COMPETITION
Spring 2005

ANSWER SHEET

Directions: You have one full hour to take this test. Scrap paper is allowed. The use of a calculator is permitted, but not stored programs on the calculator. Moreover, books, tables, and computers are not permitted. You are not expected to answer all problems. Yet, do not waste too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and nothing is deducted for a blank. There is no partial credit. Note: Some answers are asked to be in exact form. Exact answers must be expressed as fraction or a radical or in terms of π, depending on context. If not specified then answers may be given in fractional form, radical form, or as an approximation to the third decimal place as appropriate. For example, $\sqrt{7}$ or 2.646.

Name _____ College _____
Home Address _____
(include, city, state, zip code)
Name of teacher in whose class you are enrolled _____

1. B	11. B
2. 3of one sex and 1 of the other	12. C
3. 3+3+3+2+2	13. C
4. 296 2/3	14. B
5. 220	15. 894
6. 0	16. $.999^{(1/300)}$
7. 6	17. 1/49
8. None	18. 8
9. 9409	19. {5, 6, 6, 7, 8, 10}
10. 1	20. A

Correct _____ X 4 = _____

Incorrect _____ X –1 = _____

Total = _____

Math Contest Solutions

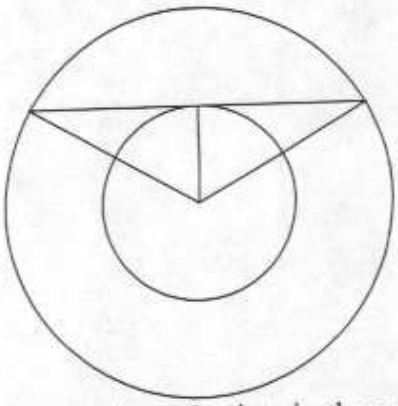
Spring 2005

1. Each of the graphs is a parabola and (-b/2a) gives the coordinate of the vertex. For a positive the parabola opens up. Thus, if t is to the right of the vertex, profits will be positive and rising. **Answer B**.
2. The probability of 2 of each sex is ${}_4C_2(1/2)^2(1/2)^2 = 3/8$. the probability of 3 of one and 1 of the other is $2[{}_4C_3(1/2)^3(1/2)] = ½$. **The answer is 3 of one sex and 1 of the other.**
3. We are looking for a collection of numbers that add to 13 and whose product is as large as possible. Trying some examples we have 11x 2 = 22, 9x2x2 = 36, 4x4x4x1 = 64, and 5x3x3x2 = 90. There is no need to have a 5 in our expression since 5 = 3 + 2 and 2 x 3 = 6. Thus, to get the maximal product we should have only 2's and 3's. Our choice is 3x3x3x2x2 = **108.**
   Answer: The partition is 3+3+3+2+2.
4. If we let x = the answer in the last step, then Dave had .3x = 22.5 so x = 75. If we let y = the number in the next to last step, then y – 7 = 75. thus, y = 82. Now proceeding correctly, 82 + 7 = 89 and 89/.3 = **296 2/3. Answer**.
5. The pattern of the blocks starting from the top down gives 1 + 3 + 6 + 10 +15. Notice the number of blocks added is the same as the number of the layer. Thus, for 10 layers, we have 1 + 3 + 6 + 10 +15 + 21 + 28 + 36 + 45 + 55 = **220**. **Answer.**
6. Notice in the product that each value is defined and cot(90°) = 0. Hence, **the answer is 0.**
7. 3 and 7 are both prime so their greatest common factor is 1. The greatest common factor of 4 and 2 is 2, the greatest common factor of 9 and 15 is 3, and the greatest common factor of 10 and 25 is 5. The greatest common factor of 6 and 6 is 6. **Answer is 6.**
8. Since each sister has one brother, there are 8 children, 7 girls and 1 boy. Adding two parents means there are 10 people, each gets their own seat. **Answer: None remain home.**
9. A number that has exactly three factors is of the form p^2 where p is prime. Thus , we are looking for the largest prime whose square is less than 10,000. **Answer:$97^2 = 9409$**.
10. $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, $S_4 = -2$, $S_5 = 3$, $S_6 = -3$, … The pattern that emerges is that if n is odd, then $S_n = (n + 1)/2$. If n is even, then $S_n = -n/2$. Thus, $S_{17} + S_{33} + S_{50} = (17 + 1)/2 + (33 + 1)/2 + 50/2 = 9 + 17 - 25 = 1$. **Answer: 1**.
11. Let x = the number voting against the first time. Then the number for the first time is 400 – x. So the margin of defeat the first time was x – (400 – x) = 2x – 400. The number voting for the second time is (12/11)x. The number voting against the second time was (400 – (12/11)x). The margin of victory on the second vote was 12x/11 – (400 – 12x/11) = 24x/11 – 400. So we have 24x/11 – 400 = 2(2x – 400). Upon solving, we have x = 220 voting against the first time. And 400 – 220 = 180 voted for it the first time. 12(220)/11 = 240, the

number voting for the second time. The difference is 240 – 180 = 60. **The answer is B.**

12. We can imagine A moving from the origin along the positive x-axis with rate x yds./min. and B moving down the y-axis from a point 500 yds. Above the origin at y yds./min. After 2 minutes A will be at (2x, 0) and B will be at (0, 500 -2y). Their respective distances from the origin are 2x and 500-2y. Thus, 2x = 500 -2y or x + y = 250. In 8 more minutes, their respective positions will be (10x, 0) and (0, 500 – 10y). At this time B would be below the x-axis so the distance of will be -(500 – 10y) = 10y – 500. The distance of A is 10x, so 10x = 10y – 500. Thus, the second equation is x – y = -50. Adding the two equations gives us x = 100 yds./min. and y = 150 yds./min. The ratio of the rate of A to the rate of B is 110/150 = 2/3. **The answer is C.**

13. If we let x = the large radius and y = the small radius, then the area between the two circles is $\pi(x^2 - y^2) = 12.5\pi$. So, $x^2 - y^2 = 12.5$. The radius meets the tangent at 90° . So if we let c be the length of half the chord, then $c^2 + y^2 = x^2$. So, $c^2 = x^2 - y^2$ = 12.5. So $c = \sqrt{(12.5)} = \sqrt{(25/2)}$. The full length of the chord is $2c = 2\sqrt{(25/2)}. = 5\sqrt{2}$. **Answer C.**



14. The number of points in the same plane equidistant form two parallel tangents to the circle and the circle is 3. There is the center of the circle and two points on either side of the circle that are on a line parallel to the two tangents and at a distance equal to the radius away from the edge of the circle. **Answer B.**
15. The number 743 written in base, b, has value $7b^2 + 4b + 3$ in base 10. The number 291 written in base 2b – 3 has value $2(2b - 3)^2 + 9(2b - 3) + 1$. These two values are the same so $7b^2 + 4b + 3 = 2(2b - 3)^2 + 9(2b - 3) + 1$. Solving this equation yields b = 11. The number is $7(11)^2 + 4(11) + 3. =$ **894. Answer.**
16. We have to decide in which order the terms are listed. If we use rational exponents, the first term is $.999^{(1/(149)(150))}$, while the last is $.999^{(1/6)}$. Since .999 < 1, the lower exponent will give us the greater value. Hence, the first term is the larger and they decrease as we go to the right. Now we need to compute the product of the first 50 terms. That is,

$$\frac{\sqrt[149]{.999}}{\sqrt[150]{.999}} \cdot \frac{\sqrt[148]{.999}}{\sqrt[149]{.999}} \cdot \frac{\sqrt[147]{.999}}{\sqrt[148]{.999}} \cdot \ldots \cdot \frac{\sqrt[100]{.999}}{\sqrt[101]{.999}}.$$

Observe that the numerator of the first fraction and the denominator of the second

$\sqrt[150]{.999}$ $.999^{(1/150)}$

17. In order to find the exact value of this number we need to rewrite the fractions in terms of integers . Thus, $\frac{(3/2)^{-11000}(7/8)^{1000}(9/4)^{4000}}{7^{1002}(3^3)^{-1000}}$. $8 = 2^3$, $9 = 3^2$, and $4 = 2^2$.

    Combining the four equations above and simplifying, we have the following.
    $\frac{3^{-11000}(1/2)^{-11000}7^{1000}(1/2)^{3000}3^{8000}(1/2)^{8000}}{7^{1002}3^{-3000}} = 7^{-2}$ = **1/49 Answer.**

18. $1 + 2 + 3 + \ldots + 6 + 7 + 8 + + \ldots. + 33 = 33(34)/2 = 561$ and $1 + 2 + \ldots + 6 = 21$. So, $7 + 8 + + \ldots. + 33 = 561 - 21 = 540$. Thus, $N = 540 + 35! + 37! + 39! + \ldots + 2205!$ Now each of the terms 5 – 20 are factors of the factorial sums since each appears in their expansion. Hence, any of the factors of N must be factors of 540. In this way the factors of N from the list are 5, 6, 9, 10, 12, 15, 18, and 20. **Answer: There are 8 that are not factors.**
19. There are 6 numbers in the list and the mode is 6 so 6 has to be the most frequent. There are at least two 6's. Now we have 5, 6, 6, _, _ , _. Since the median, 6.5, is the mean of 6 and the next number, the next number is 7. We now have 5, 6, 6, 7, _ , _. Since the mean is 7, the sum of all the numbers is 6·7 = 42. The first four numbers add to 24 and the last two must add to 18. Both numbers have to be greater than 7 and can not be equal because the mode has to be 6. The remaining two numbers are 8 and 10. **The answer is 5, 6, 6, 7, 8, 10.**
20. The graph of the first equation is a circle centered at (4, 3) with radius, 5000^{5000} while the graph of the second is a parabola with a vertex at the (0, -10000!), opening down. If $5000^{5000} > 10000!$, then the two will intersect. However, 10000! contains 5000 factors that are greater than 5000.
    Thus, $10000! > 5000^{5000}$. **The answer is 0, A.**