NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2010

(No calculators of any kind are permitted!!!)

1. If $\Psi(a,b) = \Psi(a-1,b) + b + 1$, $\Psi(a,0) = a$, and $\Psi(a,b) = \Psi(b,a)$, then what is the value of $\Psi(12,5)$.

   a) 65 b) 71 c) 77 d) 83

2. If $x - \frac{1}{x} = 3$, what is the value of $x^3 - \frac{1}{x^3}$?

3. How many real values of x satisfy the equation $\sqrt{x+\sqrt{2x-1}} + \sqrt{x-\sqrt{2x-1}} = \sqrt{2}$?

   a) 1 b) 2 c) 4 d) infinitely many

4. Everyone knows that *Peter Piper picked a peck of pickled peppers*, but how long did it take him? Peter and his friend Paul can pick a peck of peppers in 40 minutes, Peter and his other friend Mary can pick a peck of peppers in 30 minutes. If Paul and Mary can pick a peck of peppers in 60 minutes, then how long does it take Peter to pick a peck of peppers working alone?

5. Only one of the following numbers is <u>not</u> prime, which one?

   a) $12^4 + 7$ b) $12^5 + 7$ c) $12^6 + 7$ d) $12^7 + 7$

6. How many positive integers $n \le 2010$ have the property that $\lfloor \log_2(n) \rfloor$ is odd?

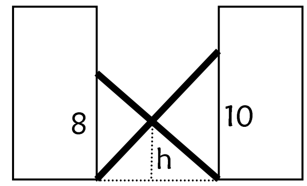
   $\lfloor x \rfloor$, called the *floor function*, denotes the largest integer less than or equal to x, e.g. $\lfloor 1.9 \rfloor = 1$.

7. $\sqrt{1{,}000{,}000} - \sqrt{999{,}999}$ is closest to

   a) $\frac{1}{1000}$ b) $\frac{1}{1500}$ c) $\frac{1}{2000}$ d) $\frac{1}{2500}$

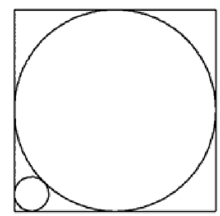
8. If an n-sided die (i.e. a die with the numbers 1 through n on it) is rolled n times, the probability of rolling exactly one of each different number of the die is about 0.04 – rounded to the nearest hundredth. Assuming each number is equally likely, what is the value of n?

9. Two ladders are propped up vertically between two buildings. The top of one ladder touches the building 10 feet from the ground, while the top of the other ladder is 8 feet from the ground, as shown. What is the height, h, at which they cross?



10. Which point on the line $y = \frac{1}{2}x - 3$ is closest to the point $(3,3)$?

11. A circle of radius 1 is inscribed in a square with a smaller circle inscribed between the circle and the lower left corner of the square, as shown. What is the radius of the smaller circle?



12. What is the exact value of $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$?

a) 3 b) $\log_{27} 33$ c) $\log_{27}(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8)$ d) $\log_{5040}(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8)$

13. The pages of a book are numbered as usual, beginning with page 1 and incremented by 1. If a total of 2010 digits were used to number all the pages, then what is the last numbered page?

14. Suppose a person can walk down a *down-escalator* in 10 seconds, and walk up the same *down-escalator* in 40 seconds. Assuming the person and the escalator both move at a constant rate, how long would it take the person to go down the escalator while standing still on it?

15. A survey of calculus students showed *exactly* 23.2% were math majors. What is the minimum number of students who could have been surveyed?

16. In how many minutes past 3 o'clock will the minute and hour hands of a clock *next* be perpendicular? Give the *exact* value.

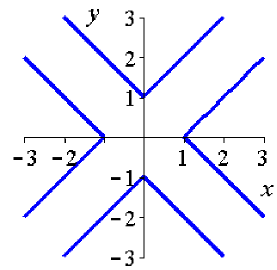
17. The following expressions arranged from smallest to largest (left to right): $\sqrt[3]{2}$, $\sqrt[5]{3}$, $\sqrt[10]{10}$, are

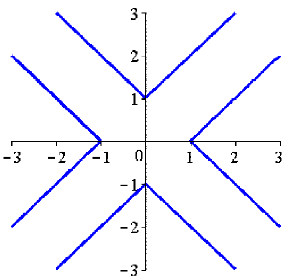
a) $\sqrt[3]{2}$, $\sqrt[5]{3}$, $\sqrt[10]{10}$ b) $\sqrt[10]{10}$, $\sqrt[3]{2}$, $\sqrt[5]{3}$ c) $\sqrt[10]{10}$, $\sqrt[5]{3}$, $\sqrt[3]{2}$ d) $\sqrt[5]{3}$, $\sqrt[10]{10}$, $\sqrt[3]{2}$

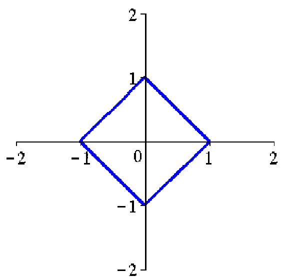
18. The *exact* value of the sum $\sum_{n=1}^{90} \sin^2(n°) = \sin^2(1°) + \sin^2(2°) + \cdots + \sin^2(89°) + \sin^2(90°)$ is

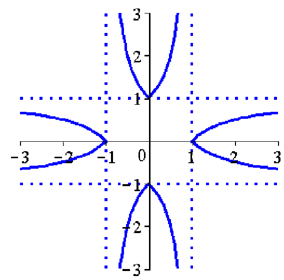
a) $\dfrac{90}{\pi}$ b) 45 c) $\dfrac{91}{2}$ d) $\dfrac{180}{\pi}$

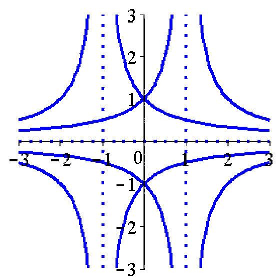
19. The graph of $f(x,y) = 1$ is shown at right. Which of the following is the graph of $\dfrac{1}{f(x,y)} = 1$?



a) 

b) 

c) 

d) 

20. At the last NYSMATYC Olympics, the 100 meter races were closely monitored. Each contestant ran in two races. Only one runner finished in the same position in both races. Ray was never last. Kim always beat Dianna. Ernie had at least one first place. Ray finished third in at least one of the races. Both Dianna and Kim had a second place. Who was it that finished both races in the same position?

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1. $\Psi(12,5)=\Psi(5,12)=\Psi(4,12)+13=\Psi(3,12)+2\cdot 13=\Psi(2,12)+3\cdot 13$

$=\Psi(1,12)+4\cdot 13=\Psi(0,12)+5\cdot 13=\Psi(12,0)+65=12+65=77\Rightarrow$ $\boxed{\text{Answer: c}}$

2. $x-\dfrac{1}{x}=3\Rightarrow\left(x-\dfrac{1}{x}\right)^2=3^2\Rightarrow x^2-1-1+\dfrac{1}{x^2}=9\Rightarrow x^2+\dfrac{1}{x^2}=11$

$\Rightarrow\left(x-\dfrac{1}{x}\right)\left(x^2+\dfrac{1}{x^2}\right)=3\cdot 11\Rightarrow x^3+\dfrac{1}{x}-x-\dfrac{1}{x^3}=33\Rightarrow x^3-\left(x-\dfrac{1}{x}\right)-\dfrac{1}{x^3}=33$

$\Rightarrow x^3-3-\dfrac{1}{x^3}=33\Rightarrow x^3-\dfrac{1}{x^3}=36$ $\boxed{\text{Answer: d}}$

3. First, we want $2x-1\geq 0$ and $x-\sqrt{2x-1}\geq 0$. Thus, $x\geq\dfrac{1}{2}$ and

$x^2\geq 2x-1\Rightarrow x^2-2x+1\geq 0\Rightarrow(x-1)^2\geq 0$, which is true for all real x. Thus, $x\geq\dfrac{1}{2}$.

Now solving the equation…square both sides: $\left(\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}\right)^2=\left(\sqrt{2}\right)^2$

$\Rightarrow x+\sqrt{2x-1}+2\sqrt{\left(x+\sqrt{2x-1}\right)\left(x-\sqrt{2x-1}\right)}+x-\sqrt{2x-1}=2$

$\Rightarrow x+2\sqrt{x^2-(2x-1)}+x=2\Rightarrow\sqrt{x^2-2x+1}=1-x\Rightarrow\sqrt{(1-x)^2}=1-x\Rightarrow|1-x|=1-x\Rightarrow x\leq 1.$

Hence, the equation is satisfied for $\dfrac{1}{2}\leq x\leq 1$, which gives <u>infinitely many solutions</u>.

(Actually, the equation is satisfied for all real $x\leq 1$!!!) $\boxed{\text{Answer: d}}$

4. Let x be the rate at which Peter works, y be the rate at which Paul works, and z be the rate at which Mary works — all rates in pecks/minute. Since $Work=rate\cdot time$, and *Work* being additive, we get: $1=40(x+y)$, $1=30(x+z)$, and $1=60(y+z)$. These give $x+y=\frac{1}{40}$, $x+z=\frac{1}{30}$, and $y+z=\frac{1}{60}$. Solving for x gives $x=\frac{1}{48}$. Thus, Peter working alone picks $\frac{1}{48}$ of a peck per minute. Which means it would take him 48 minutes working alone. $\boxed{\text{Answer: 48 minutes}}$

5. The unit's digit of 12^4 is 6, so the unit's digit of 12^4+7 is 3. The unit's digit of 12^5 is 2, so the unit's digit of 12^5+7 is 9. The unit's digit of 12^6 is 4, so the unit's digit of 12^6+7 is 1. The unit's digit of 12^7 is 8, so the unit's digit of 12^7+7 is 5. Thus, 12^7+7 cannot be prime, since any integer with a 5 in the unit's digit has 5 as a factor. $\boxed{\text{Answer: d}}$

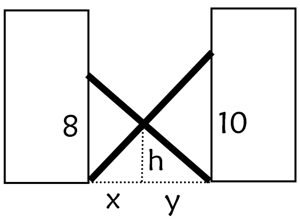
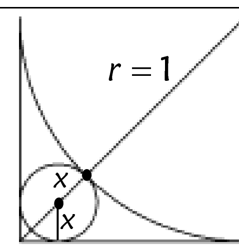
6. $\log_2(1)=0$, $\log_2(2)=1$, $\log_2(4)=2$, $\log_2(8)=3$, $\log_2(16)=4$, $\log_2(32)=5$, $\log_2(64)=6$,

$\log_2(128)=7$, $\log_2(256)=8$, $\log_2(512)=9$, $\log_2(1024)=10$, $\log_2(2048)=11$.

Thus, $\lfloor\log_2(n)\rfloor$ is odd for:

$$n=\underbrace{2,3,8}_{3},\underbrace{9,\cdots,15}_{7},\underbrace{32}_{1},\underbrace{33,\cdots,63}_{31},\underbrace{128}_{1},\underbrace{129,\cdots,255}_{127},\underbrace{512}_{1},\underbrace{513,\cdots,1023}_{511}.$$

Which gives $3+7+1+31+1+127+1+511=$ $\boxed{\text{682 valaues}}$.

7.	$\sqrt{1,000,000}-\sqrt{999,999}=\left(\sqrt{1,000,000}-\sqrt{999,999}\right)\left(\frac{\sqrt{1,000,000}+\sqrt{999,999}}{\sqrt{1,000,000}+\sqrt{999,999}}\right)$ $=\frac{1,000,000-999,999}{\sqrt{1,000,000}+\sqrt{999,999}}=\frac{1}{\sqrt{1,000,000}+\sqrt{999,999}}\approx\frac{1}{1000+1000}=\frac{1}{2000}\Rightarrow$ $\boxed{\text{Answer: c}}$
8.	The probability of rolling any one of the n-numbers is $\frac{1}{n}$. Thus, the probability of rolling all n of them on n rolls is $P(n)=\left(\frac{1}{n}\right)^n\cdot n!=\frac{n!}{n^n}$ (there are $n!$ outcomes). $P(3)=\frac{3!}{3^3}=\frac{3\cdot2\cdot1}{3\cdot3\cdot3}=\frac{2}{9}=0.2\overline{2}$, $P(4)=\frac{4!}{4^4}=\frac{4\cdot3\cdot2\cdot1}{4\cdot4\cdot4\cdot4}=\frac{3}{32}\approx0.1$, $P(5)=\frac{5!}{5^5}=\frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot5\cdot5\cdot5\cdot5}=\frac{24}{625}\approx\frac{25}{625}=\frac{1}{25}=0.04\Rightarrow\boxed{n=5}$
9.	 By similar triangles: $\frac{h}{y}=\frac{8}{x+y}$ and $\frac{h}{x}=\frac{10}{x+y}$ $\Rightarrow h(x+y)=8y$ and $h(x+y)=10x\Rightarrow 8y=10x$ $\Rightarrow y=\frac{5}{4}x$, so that $\frac{h}{\frac{5}{4}x}=\frac{8}{x+\frac{5}{4}x}\Rightarrow\frac{h}{\frac{5}{4}x}=\frac{8}{\frac{9}{4}x}\Rightarrow\frac{9}{4}h=10$ $\Rightarrow\boxed{h=\frac{40}{9}ft=4.4\overline{4}\,ft}$
10.	The closest point on the line will form a perpendicular segment between the two points. The line perpendicular to $y=\frac{1}{2}x-3$ that contains (3, 3) is $y-3=-2(x-3)\Rightarrow y=-2x+9$. The closest point will be where the two lines intersect: $\frac{1}{2}x-3=-2x+9\Rightarrow\frac{5}{2}x=12\Rightarrow x=\frac{24}{5}=4.8$ $\Rightarrow y=-2\left(\frac{24}{5}\right)+9=-\frac{3}{5}=-0.6$. Thus, the closest point is $\boxed{\left(\frac{24}{5},-\frac{3}{5}\right)=(4.8,-0.6)}$.
11.	Let x = the radius of the smaller circle. The distance from the center of the large circle to the lower left corner of the square is $\sqrt{2}$, so the distance from the point of tangency of the two circles and the lower left corner of the square is $\sqrt{2}-1$. The distance from the center of the small circle to the lower left corner is $\sqrt{2}x$, so the distance from the point of tangency of the two circles to the lower left corner is also $\sqrt{2}x+x$. Thus, $\sqrt{2}x+x=\sqrt{2}-1$  which gives: $x=\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{\left(\sqrt{2}-1\right)^2}{2-1}=\frac{2-2\sqrt{2}+1}{1}=3-2\sqrt{2}$. $\boxed{\text{Answer: }3-2\sqrt{2}\text{ or }\frac{\sqrt{2}-1}{\sqrt{2}+1}}$
12.	$\left(\log_2 3\right)\left(\log_3 4\right)\left(\log_4 5\right)\left(\log_5 6\right)\left(\log_6 7\right)\left(\log_7 8\right)=\left(\frac{\log 3}{\log 2}\right)\left(\frac{\log 4}{\log 3}\right)\left(\frac{\log 5}{\log 4}\right)\left(\frac{\log 6}{\log 5}\right)\left(\frac{\log 7}{\log 6}\right)\left(\frac{\log 8}{\log 7}\right)$ $=\left(\frac{1}{\log 2}\right)\left(\frac{\log 8}{1}\right)=\frac{\log 8}{\log 2}=\log_2 8=3$ $\boxed{\text{Answer: a}}$
13.	Pages 1 through 9 contribute 1 digit each, for a total of 9 digits; pages 10 through 99 contribute 2 digits each, for a total of 2(90)=180 digits; pages 100 through 199 contribute 3 digits each, for a total of 3(100)=300 digits. Thus, pages 1 through 199 use 9+180+300=489 digits. We need $2010-489=1521$ more digits. Five more groups of 300 digits (for pages 200 through 699) give us 1500 more digits, for a total of 1500+489=1989 digits used for pages 1 through 699. Now we need $2010-1989=21$ more digits. Since $21\div3=7$, we need only 7 more 3-digit numbers – being 700 through 706. $\boxed{\text{Answer: 706}}$

14.	Let x = the time required for the person to walk up (or down) the escalator if it were stopped, and y = the time required for the escalator to move a person down if the person stands still on it. Thus, $\frac{1}{x}$ and $\frac{1}{y}$ are the rates for each and we get the following equations: $\left(\frac{1}{x}+\frac{1}{y}\right)\cdot 10=1$ and $\left(\frac{1}{x}-\frac{1}{y}\right)\cdot 40=1$, using *distance* = *rate* · *time*. These give: $\frac{1}{x}+\frac{1}{y}=\frac{1}{10}$ and $\frac{1}{x}-\frac{1}{y}=\frac{1}{40}$. Subtracting the two equations yields: $\frac{2}{y}=\frac{1}{10}-\frac{1}{40}\Rightarrow\frac{2}{y}=\frac{3}{40}\Rightarrow y=\frac{80}{3}$. $\boxed{\text{Answer: } \frac{80}{3}=26.\overline{6} \text{ seconds}}$
15.	$23.2\%=\dfrac{232}{1000}=\dfrac{2^3\cdot 29}{2^3\cdot 5^3}=\dfrac{29}{125}$ (in simplest form). Thus, $\boxed{125}$ is the smallest possible number.
16.	The minute hand makes a full rotation every 60 minutes, so it rotates at a rate of 360° per 60 minutes, or 6° per minute. The hour hand rotates one-twelfth of a full rotation every 60 minutes, so it rotates at a rate of 30° per 60 minutes, or (½)° per minute. Starting at 3 o'clock, the minute hand's angular displacement is $0+6t=6t$ degrees and the hour hand's is $90+\frac{1}{2}t$ degrees, t minutes later. We need to solve $6t-\left(90+\frac{1}{2}t\right)=90$, when the minute hand will be 90° degrees ahead of the hour hand. This yields $\frac{11}{2}t=180\Rightarrow t=\frac{360}{11}\Rightarrow$ $\boxed{\frac{360}{11} \text{ or } 32\frac{8}{11} \text{ minutes}}$.
17.	The expressions can be rewritten as: $2^{\frac{1}{3}}$, $3^{\frac{1}{5}}$, and $10^{\frac{1}{10}}$. Since all four quantities are positive, raising them to the same positive power preserves their relative positions on the number line. Thirty is the smallest common power that will transform each to a whole number. This yields: $\left(2^{\frac{1}{3}}\right)^{30}=2^{10}=1024$, $\left(3^{\frac{1}{5}}\right)^{30}=3^6=729$, and $\left(10^{\frac{1}{10}}\right)^{30}=10^3=1000$. $\Rightarrow 729<1000<1024$. Thus, we obtain $\sqrt[5]{3}<\sqrt[10]{10}<\sqrt[3]{2}$. $\boxed{\text{Answer: d}}$
18.	Let $x=\sum_{n=1}^{90}\sin^2(n^\circ)=\sum_{n=0}^{90}\sin^2(n^\circ)=\sin^2(0^\circ)+\sin^2(1^\circ)+\cdots+\sin^2(89^\circ)+\sin^2(90^\circ)$ $=\sum_{n=0}^{90}\left[1-\cos^2(n^\circ)\right]=\sum_{n=0}^{90}1-\sum_{n=0}^{90}\cos^2(n^\circ)$ $=91-\left[\cos^2(0^\circ)+\cos^2(1^\circ)+\cdots+\cos^2(89^\circ)+\cos^2(90^\circ)\right]$ $=91-\left[\sin^2(90^\circ)+\sin^2(89^\circ)+\cdots+\sin^2(1^\circ)+\sin^2(0^\circ)\right]$, since $\cos(n^\circ)=\sin(90^\circ-n^\circ)$ $\Rightarrow x=91-x\Rightarrow 2x=91\Rightarrow x=\dfrac{91}{2}\Rightarrow$ $\boxed{\text{Answer: c}}$
19.	$\dfrac{1}{f(x,y)}=1\Rightarrow f(x,y)=\dfrac{1}{1}\Rightarrow f(x,y)=1\Rightarrow$ The same graph. $\Rightarrow$ $\boxed{\text{Answer: a}}$
20.	Race 1 (or 2): Kim, Dianna, Ray, Ernie. Race 2 (or 1): Ernie, Kim, Ray, Dianna. Since Kim always beat Dianna and Dianna had a second place, one race must have been Kim first and Dianna second. Ernie therefore won the other race with Kim second. Since only one runner finished in the same place in both races, this must have been Ray in third. $\boxed{\text{Answer: Ray}}$