New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2011

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. If $f(x)-2f(2012-x)=x$, what is the numerical value of $f(2011)$?

2. How many real values of x satisfy the equation $\left(x^2+7x+11\right)^{x^2+3x-10}=1$?

   a) 2 b) 3 c) 4 d) 5

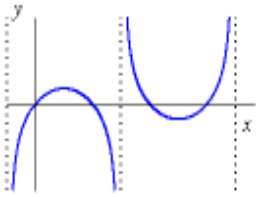
3. The notation $a|b$ is read *as a divides b* (i.e. b is a multiple of a). What is the largest integer value of n so that $5^n|200!$? Note: $200!$ is 200 *factorial* and defined as $200!=200\cdot199\cdot198\cdot\ldots\cdot3\cdot2\cdot1$

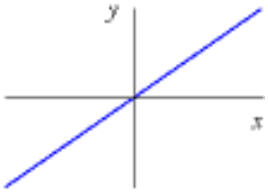
   a) 48 b) 49 c) 89 d) 90

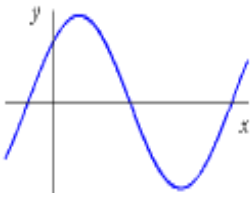
4. A group of friends decided to rent a boat for a few hours and share equally in the cost. If they had one more person, the cost per person would decrease by \$5. If they had three more people, the cost per person would decrease by \$12. If the cost of the boat is the same for any number of people, then how many people are in the group?

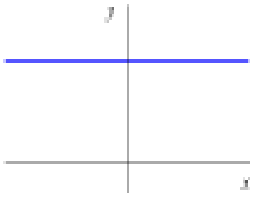
5. The equation $ax^4-7x^3+8x^2-7x+a=0$ has two distinct real roots and two complex roots, with one of the real roots being $x=2011$, for some real value of a. What is the other real root?

6. Which of the following is the graph of $f(x)=\dfrac{1}{\cos^{-1}(x)+\sin^{-1}(x)}$?

a) 

b) 

c) 

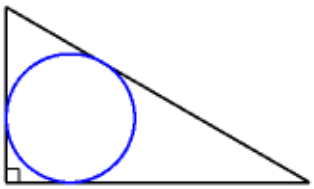
d) 

7. What is $\sin(15^\circ)+\sin(75^\circ)$?

a) 1 b) $\frac{\sqrt{6}}{2}$ c) $\frac{1+\sqrt{3}}{2}$ d) $\frac{\sqrt{2}+\sqrt{3}}{2}$

8. A regular polygon (i.e. a polygon with congruent sides and interior angles) has 54 diagonals. How many sides does this polygon have?

9. A 30°-60°-90° (right) triangle has a circle of radius 1 inscribed in it, as shown. In simplest form, what is the area of the triangle?



10. Point A is 1 unit from Point B, while Point B is 1 unit from Point C. What is the probability that Point A is closer to Point C than it is to Point B? Note: All points are on the same plane.

11. How many points with integer coordinates are on the line $y=\pi x-\frac{22}{7}$?

a) 0 b) 1 c) 2 d) infinitely many

12. $\sqrt{9+4\sqrt{5}}=?$

a) $3+\frac{1}{2}\sqrt{5}$ b) $2+\sqrt{5}$ c) $3\sqrt[4]{5}$ d) $3+\sqrt[4]{5}$

13. What value of $x>1$ satisfies the equation $\log_2\left[\log_4(x)\right]=\log_8\left[\log_2(x)\right]$?

14. Three valves A, B, and C, when open, release water into a tank at their own constant rate. When all three valves are open the tank fills in 1 hour, with only valves A and C open the tank fills in 90 minutes, and with only valves B and C open it takes 2 hours. How long would it take to fill the tank with only valves A and B open?

15. If $S(n)=\sum_{k=2}^{n}\frac{1}{\log_k(n!)}$, what is the value of $S(2011)$?

a) $\frac{2010}{2011}$ b) 1 c) $\frac{2011}{2010}$ d) $\frac{2011}{\log_{2011}(2011!)}$

16. A check is written in the amount of x dollars and y cents, where x and y are both two-digit numbers. The bank teller mistakenly cashes it for y dollars and x cents, giving the patron more than the actual check amount. Which of the following *cannot* be the amount of over payment?
a) \$1.98 b) \$4.95 c) \$9.72 d) \$10.89

17. Two equally matched teams play a series of five games, the first to win three games is declared the winner. If Team A has won the first game, what is the probability that Team A will win the series?

18. The lines given by $y = x$ and $y = 3x$ form an acute angle in the first quadrant. What is the slope of the line that bisects that angle?
a) $\sqrt{2}$ b) $\frac{1+\sqrt{5}}{2}$ c) $\sqrt{3}$ d) 2

19. In a group of five people, the sums of the ages of each group of four of them are 138, 144, 151, 153, and 158. What is the age of the youngest person?
a) 26 b) 28 c) 29 d) 31

20. Which of the following statements are true?

I. Exactly one of these statements is true.
II. Exactly two of these statements are true.
III. Exactly three of these statements are false.
IV. Exactly two of these statements are false.

a) I only b) II only c) I and III d) II and IV

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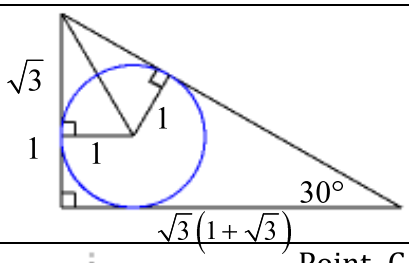
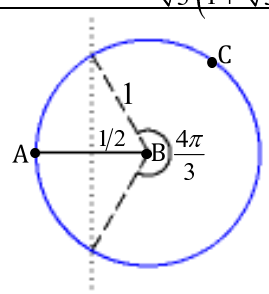
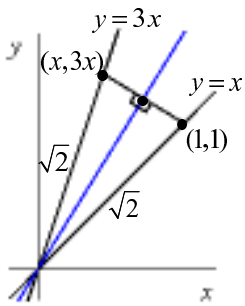
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2011 ~ Solutions

1.	Letting $x=1$ gives: $f(1)-2f(2011)=1$, and letting $x=2011$ gives: $f(2011)-2f(1)=2011$. Solving these two linear equations for $f(2011)$, gives $f(2011)=-671$. Answer: -671
2.	$\left(x^2+7x+11\right)^{x^2+3x-10}=1$ will be true only when $x^2+7x+11=1$, $x^2+3x-10=0$ with $x^2+7x+11\neq 0$, or when $x^2+7x+11=-1$ and $x^2+3x-10$ even. $x^2+7x+11=1 \Rightarrow x^2+7x+10=0 \Rightarrow (x+5)(x+2)=0 \Rightarrow x=-5, x=-2.$ $x^2+3x-10=0 \Rightarrow (x+5)(x-2)=0 \Rightarrow x=-5, x=2.$ $x^2+7x+11=-1 \Rightarrow (x+4)(x+3)=0 \Rightarrow x=-4, x=-3$ in either case $x^2+3x-10$ is even. Thus, $x\in\{-5,-4,-3-2,2\}$. Answer: d
3.	5 will divide every multiple of 5 once, every multiple of $5^2=25$ twice, every multiple of $5^3=125$ three times. $200\div 5=40$, $200\div 25=8$, and 125 divides 200 only once. Thus, there are $40+8+1=49$ fives that will factor out of $200!$. Answer: b
4.	Let x represent the number of people in the group, and C be the cost for the rental. Thus, we get the equations: $\frac{C}{x+1}=\frac{C}{x}-5$ and $\frac{C}{x+3}=\frac{C}{x}-12$. Solving gives $x=7$ (and $C=280$). Answer: 7
5.	Noticing the symmetry in the coefficients of the polynomial $ax^4-7x^3+8x^2-7x+a$, divide the equation $ax^4-7x^3+8x^2-7x+a=0$ by x^4 to obtain $a-7x^{-1}+8x^{-2}-7x^{-3}+ax^{-4}=0$, for non-zero x. This new equation can be rewritten as $a-7\left(x^{-1}\right)^1+8\left(x^{-1}\right)^2-7\left(x^{-1}\right)^3+a\left(x^{-1}\right)^4=0$ or $a\left(x^{-1}\right)^4-7\left(x^{-1}\right)^3+8\left(x^{-1}\right)^2-7\left(x^{-1}\right)^1+a=0$. Hence, if x solves the original equation, then so will x^{-1}. Thus, $x=2011^{-1}$ must be the other real root. Answer: 2011^{-1} or $\frac{1}{2011}$
6.	$\sin(t)=\cos\left(\frac{\pi}{2}-t\right)\Rightarrow\cos^{-1}\left(\sin(t)\right)=\frac{\pi}{2}-t\Rightarrow\cos^{-1}\left(\sin(t)\right)+t=\frac{\pi}{2}$, now let $x=\sin(t)$, so that $t=\sin^{-1}(x)\Rightarrow\cos^{-1}(x)+\sin^{-1}(x)=\frac{\pi}{2}\Rightarrow f(x)=\frac{1}{\left(\frac{\pi}{2}\right)}=\frac{2}{\pi}$, which is a horizontal line. Answer: d
7.	Let $x=\sin(15^\circ)+\sin(75^\circ)$, but $\sin(15^\circ)=\cos(75^\circ)$. Thus, $x=\cos(75^\circ)+\sin(75^\circ)$. Squaring: $x^2=\cos^2(75^\circ)+2\cos(75^\circ)\sin(75^\circ)+\sin^2(75^\circ)=\underbrace{\cos^2(75^\circ)+\sin^2(75^\circ)}_{1}+\underbrace{2\cos(75^\circ)\sin(75^\circ)}_{\sin(2\cdot 75^\circ),\text{ double angle formula}}$ $x^2=1+\sin(150^\circ)=1+\sin(30^\circ)=1+\frac{1}{2} \Rightarrow x^2=\frac{3}{2} \Rightarrow x=\sqrt{\frac{3}{2}}$, taking the positive square root, since $x>0$. $x=\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{2}{2}}=\frac{\sqrt{6}}{2}$. Answer: b

8.	A regular polygon with n sides has n vertices. Each of the n vertices can form a diagonal with all vertices except itself and the two adjacent ones. Thus, there are $n(n-3)$ diagonals that can be formed. However, that counts each diagonal twice (since it counts the one from vertex A to vertex B and from vertex B to vertex A). Hence, there are $\frac{1}{2}n(n-3)$ diagonals. Solving $\frac{1}{2}n(n-3)=54$, gives $n^2-3n-108=0 \Rightarrow (n+9)(n-12)=0 \Rightarrow n=-9,\ 12$. Taking the positive answer... Answer: 12
9.	 The two newly formed (smaller) triangles are both 30°-60°-90° (right) triangles. Which makes the height of the large triangle $1+\sqrt{3}$, thus making its base $\sqrt{3}(1+\sqrt{3})$. Hence, the area is $\frac{1}{2}\cdot\sqrt{3}(1+\sqrt{3})\cdot(1+\sqrt{3})=3+2\sqrt{3}$. Answer: $3+2\sqrt{3}$
10.	 Point C can be anywhere on the circumference of the circle of radius 1 centered at Point B. The vertical dotted-line is the perpendicular bisector of segment AB. Thus, if Point C is on any part of the circle to the left of the vertical line, it will be closer to Point A. The angle that subtends the arc closer to Point A is 120° or $\frac{2\pi}{3}$ radians, which has a length of $\frac{2\pi}{3}\cdot 1=\frac{2\pi}{3}$. The circumference of the circle is 2π. Hence, the probability Point C is closer to Point A is $\left(\frac{2\pi}{3}\right)\div(2\pi)=\frac{1}{3}$. Answer: $\frac{1}{3}$
11.	Suppose (a,b) is a point on the line with a and b integers. Thus, $b=\pi a-\frac{22}{7}\Rightarrow b+\frac{22}{7}=\pi a$, and $\pi a=\frac{7b+22}{7}\Rightarrow \pi=\frac{7b+22}{7a}$, $a\neq 0$. Which means that π can be expressed as the ratio of two integers, which is impossible, since π is irrational! If $a=0$, then $b=-\frac{22}{7}$, which is not an integer. Hence, there are no points on the line with integer coordinates. Answer: a
12.	Let $\sqrt{9+4\sqrt{5}}=a+\sqrt{b}\Rightarrow\left(\sqrt{9+4\sqrt{5}}\right)^2=\left(a+\sqrt{b}\right)^2\Rightarrow 9+4\sqrt{5}=a^2+b+2a\sqrt{b}$, which gives $a^2+b=9$ and $2a\sqrt{b}=4\sqrt{5}\Rightarrow b=5$ and $a=2\Rightarrow\sqrt{9+4\sqrt{5}}=2+\sqrt{5}$ Answer: b
13.	$\log_2\left[\log_4(x)\right]=\log_8\left[\log_2(x)\right]\Rightarrow\log_2\left[\log_4(x)\right]=\frac{\log_2\left[\log_2(x)\right]}{\log_2 8}\Rightarrow\log_2\left[\log_4(x)\right]=\frac{\log_2\left[\log_2(x)\right]}{3}$ $\Rightarrow 3\log_2\left[\log_4(x)\right]=\log_2\left[\log_2(x)\right]\Rightarrow\log_2\left[\left(\log_4(x)\right)^3\right]=\log_2\left[\log_2(x)\right]\Rightarrow\left(\log_4(x)\right)^3=\log_2(x)$ $\Rightarrow\left(\frac{\log_2(x)}{\log_2(4)}\right)^3=\log_2(x)\Rightarrow\left(\frac{\log_2(x)}{2}\right)^3=\log_2(x)\Rightarrow\frac{\left(\log_2(x)\right)^3}{8}=\log_2(x)\Rightarrow\left(\log_2(x)\right)^3=8\log_2(x)$ $\Rightarrow\left(\log_2(x)\right)^3-8\log_2(x)=0\Rightarrow\log_2(x)\left[\left(\log_2(x)\right)^2-8\right]=0\Rightarrow\log_2(x)=0$ or $\left(\log_2(x)\right)^2-8=0$ Since $x>1$, $\left(\log_2(x)\right)^2-8=0\Rightarrow\log_2(x)=\sqrt{8}\Rightarrow x=2^{\sqrt{8}}=2^{2\sqrt{2}}=4^{\sqrt{2}}$ Answer: $2^{\sqrt{8}}$ or $2^{2\sqrt{2}}$ or $4^{\sqrt{2}}$
14.	Let a = the rate at which water flows from Valve A, b = the rate at which water flows from Valve B, and c = the rate at which water flows from Valve C. Hence, we can write the following equations from the given information (using Work = Rate · Time): ❶ $(a+b+c)\cdot 1=1$, ❷ $(a+c)\cdot\frac{3}{2}=1$, and ❸ $(b+c)\cdot 2=1$, using time in hours and Work = 1 meaning one full tank. No need to solve for a, b, and c, just solve for c in terms of a and b from equations ❷ and ❸, then substitute into ❶. Doing this, we obtain: $c=\frac{7}{12}-\frac{1}{2}a-\frac{1}{2}b$, now substituting this into ❶ gives: $\frac{1}{2}a+\frac{1}{2}b=1-\frac{7}{12}$ $\Rightarrow\frac{1}{2}(a+b)=\frac{5}{12}\Rightarrow\frac{12}{10}(a+b)=1\Rightarrow\frac{6}{5}(a+b)=1$ Answer: $\frac{6}{5}$ hrs. or 1.2 hrs. or 72 minutes

15.	$S(n)=\sum_{k=2}^{n}\frac{1}{\log_k(n!)}=\sum_{k=2}^{n}\frac{1}{\left[\frac{\log_n(n!)}{\log_n(k)}\right]}=\sum_{k=2}^{n}\frac{\log_n(k)}{\log_n(n!)}=\frac{1}{\log_n(n!)}\sum_{k=2}^{n}\log_n(k)$ $=\frac{1}{\log_n(n!)}\left[\log_n(2)+\log_n(3)+\log_n(4)+\ldots+\log_n(n-2)+\log_n(n-1)+\log_n(n)\right]$ $=\frac{1}{\log_n(n!)}\left[\log_n\left(2\cdot3\cdot4\cdot\ldots\cdot(n-2)\cdot(n-1)\cdot n\right)\right]=\frac{1}{\log_n(n!)}\left[\log_n(n!)\right]=1$ Answer: b
16.	The check amount is $100x+y$ cents. It was cashed for $100y+x$, with a difference of $(100x+y)-(100y+x)=100(x-y)+(y-x)=100(x-y)-(x-y)=99(x-y)$. Hence, the amount of over payment (in cents) must be a multiple of 99. $198=2\cdot99$, $1089=11\cdot99$, and $495=5\cdot99$, only 972 is not a multiple of 99. Answer: c
17.	If the teams are equally matched, then the probability of each winning any one game is ½. In order for Team A to win, the team would have to win either the next 2 games, or 2 of the next 3 or 4 games with the final game being a win. We can symbolize this as: AA, ABA, BAA, ABBA, BABA, BBAA, which yields the following probabilities: $\left(\frac{1}{2}\right)^2,\left(\frac{1}{2}\right)^3,\left(\frac{1}{2}\right)^3,\left(\frac{1}{2}\right)^4,\left(\frac{1}{2}\right)^4,\left(\frac{1}{2}\right)^4$. Summing these gives: $\left(\frac{1}{2}\right)^2\left[1+\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]=\frac{1}{4}\left[2+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right]=\frac{1}{4}\left(\frac{11}{4}\right)=\frac{11}{16}$ Answer: $\frac{11}{16}$
18.	 Draw a line segment that is perpendicular to the bisector to create two congruent right triangles, as shown. Taking the point $(1,1)$ on the line $y=x$ yields a length of $\sqrt{2}$ for the hypotenuse of the triangle. Thus, the other hypotenuse must also have a length of $\sqrt{2}$. Now determine the point on the line $y=3x$ labeled $(x,3x)$: $x^2+(3x)^2=\left(\sqrt{2}\right)^2\Rightarrow10x^2=2\Rightarrow x^2=\frac{1}{5}$ $\Rightarrow x=\sqrt{\frac{1}{5}}=\frac{\sqrt{5}}{5}$, taking the positive value for x, which makes the y-value $3\frac{\sqrt{5}}{5}$. The midpoint of the segment, which is on the bisector, can now be found: $\left(\frac{1+\sqrt{5}/5}{2},\frac{1+3\sqrt{5}/5}{2}\right)$. So the slope of the bisector is: $\frac{\left(\frac{1+3\sqrt{5}/5}{2}-0\right)}{\left(\frac{1+\sqrt{5}/5}{2}-0\right)}=\frac{1+\frac{3\sqrt{5}}{5}}{1+\frac{\sqrt{5}}{5}}=\frac{5+3\sqrt{5}}{5+\sqrt{5}}=\frac{5+3\sqrt{5}}{5+\sqrt{5}}\cdot\frac{5-\sqrt{5}}{5-\sqrt{5}}=\frac{25+10\sqrt{5}-15}{25-5}=\frac{1+\sqrt{5}}{2}$ Answer: b
19.	Adding all sums of groups of four: $138+144+151+153+158=744$. This sum includes each age exactly four times, hence $744\div4=186$ is the sum of all five ages. The largest sum (158) of four of the ages must omit the lowest age. Therefore, the youngest is $186-158=28$. Answer: b
20.	If Statement I were true, then exactly three would be false, making Statement III also true. Hence, Statements I and III cannot be true. Statements II and IV are consistent. Answer: d