New York State Mathematics
Association of Two-Year Colleges

Math League Contest ~ Spring 2013

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. If $2(1-x)f(1-x)-xf(x)=2013x^2-2016x+2$, then express $f(x)$ explicitly as a function of x only.

2. The pattern in the expression $3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\ldots}}}}$ continues indefinitely, i.e. the 3 and 6 alternate (after the first "3") in what is called a *continued fraction*. To what value does it converge?

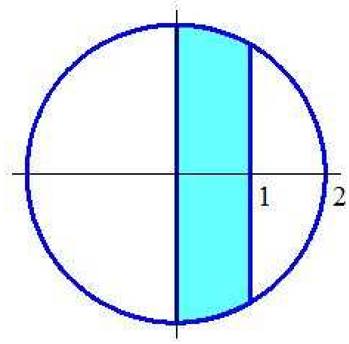
   a) $\sqrt{10}$ b) $\frac{29}{9}$ c) $\sqrt{11}$ d) $\frac{10}{3}$

3. What is the area of a triangle with sides of length $\sqrt{3}$, $\sqrt{4}$, and $\sqrt{5}$?

   a) $\frac{\sqrt{11}}{2}$ b) $\sqrt{3}$ c) $\frac{\sqrt{15}}{2}$ d) $\sqrt{6}$

4. The fast-food restaurant MathDonald's sells chicken nuggets only in boxes of size 5 and 8. What is the largest number of chicken nuggets that *cannot* be purchased? For example, 21 nuggets *can* be purchased (1 box of 5 nuggets and 2 boxes of 8 nuggets), but 22 nuggets *cannot*.

5. The diagram shows a circle of radius 2 centered at the origin. The region bounded by the circle, the y-axis, and the line $x = 1$ has been shaded. What is the area of the shaded region?



6. If $P(x)$ is a polynomial such that $P(2x+3)=8x^2+18x+10$, then express $P(x)$ in terms of x.

7. $\sqrt[10]{2},\ \sqrt[50]{213},\ \sqrt[100]{2013}$ arranged from smallest to largest is

a) $\sqrt[10]{2},\ \sqrt[50]{213},\ \sqrt[100]{2013}$ b) $\sqrt[100]{2013},\ \sqrt[50]{213},\ \sqrt[10]{2}$

c) $\sqrt[10]{2},\ \sqrt[100]{2013},\ \sqrt[50]{213}$ d) $\sqrt[50]{213},\ \sqrt[10]{2},\ \sqrt[100]{2013}$

8. How many trailing zeros are there in 2013! ? Note: $n!=n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 2\cdot 1$
For example, $15!=1{,}307{,}674{,}368{,}000$ and thus has 3 trailing zeros.

9. One leg of a right triangle has a length of 15. If the length of all three sides are integers, then how many different triangles are possible?

a) two b) three c) four d) more than four

10. If there are seven people at a gathering, what is probability they were born on different days of the week?

a) $\left(\frac{1}{7}\right)^7$ b) $\frac{6}{7}\cdot\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}\cdot\frac{2}{7}\cdot\frac{1}{7}$ c) $\frac{1}{7}$ d) $\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\frac{1}{7^4}+\frac{1}{7^5}+\frac{1}{7^6}+\frac{1}{7^7}$

11. Alphametic puzzles are arithmetic problems which involve words where each letter represents a unique digit that makes the equation true. If **I** represents the digit 0 and **N** represents the digit 4, then what digit is represented by **E**?

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  OBAMA
+ BIDEN
 WINNER
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12. Which of the following functions is *not* odd? Recall: The graph of an odd function is symmetric about the origin.

a) $\arccos(\cos(x))$ b) $\arcsin(x)$ c) $\cos(\cos(x))\cdot\sin(\sin(x))$ d) $\ln\left(\sqrt{x^2+1}-x\right)$

13. If $\sum_{n=1}^{\infty}\sin^n(x)=2013$, for an acute angle x, then what is the value of $\cos(x)$?

Note: $\sum_{n=1}^{\infty}\sin^n(x)=\sin(x)+\sin^2(x)+\sin^3(x)+\sin^4(x)+\ldots$

a) $\frac{1}{\sqrt{4027}}$ b) $\frac{\sqrt{2013}}{2014}$ c) $\frac{1}{\sqrt{2013}}$ d) $\frac{\sqrt{4027}}{2014}$

14. A certain school has a foreign language requirement. All students must take at least one of the following languages: French, Italian, Russian, or Spanish. However, no one may take more than two of them. If there are 150 students taking French, 175 taking Italian, 200 taking Russian, 225 taking Spanish, and 100 who are taking exactly two languages, then how many students are in the school?
a) 550 b) 600 c) 650 d) It cannot be determined from the given information.

15. Three people play a game where the loser must double the money of the other two. After three games, each has lost only once, and each has \$32.00. How much did the person who lost first have at the start of the beginning of all the games?

16. If x and y are positive real numbers with $2\log(x-6y)=\log(x)+\log(y)$, then what is the numerical value of $\frac{x}{y}$?

17. Give a polynomial, $P(x)$, of smallest degree with integer coefficients that has $\sqrt{3}-\sqrt{2}$ as a root.

18. What is the radius of the circle that passes through the points: $(0,0)$, $(1,1)$ and $(11,1)$?
a) $\frac{\sqrt{122}}{2}$ b) $\sqrt{61}$ c) $\sqrt{122}$ d) $2\sqrt{31}$

19. A certain party consisted of only married couples. Everyone shook hands with everyone else exactly once, except no one shook the hand of his or her spouse, or with himself or herself. If there were a total of 264 handshakes, then how many couples attended the party?

20. An explorer wishes to cross a barren desert that requires 6 days to cross, but one man can only carry enough food for 4 days. What is the *fewest number of other men* required to help carry enough food for him to cross?
a) 2 b) 3 c) 4 d) It does not matter, it cannot be done.

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New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2013 ~ Solutions

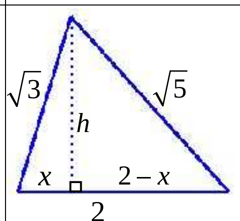
1. Given: ❶ $2(1-x)f(1-x)-xf(x)=2013x^2-2016x+2$, replacing "$x$" with "$1-x$" gives ❷ $2xf(x)-(1-x)f(1-x)=2013x^2-2010x-1$. Now solve for $f(x)$. Doubling equation ❷ and adding it to equation ❶, to eliminate $f(1-x)$, gives $3xf(x)=6039x^2-6036x$. Dividing by $3x$ yields $f(x)=2013x-2012$.
Answer: $f(x)=2013x-2012$

2. Let $x=3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\dots}}}}$, then $x+3=6+\cfrac{1}{3+\underbrace{\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\dots}}}}_{x+3}}$ $\Rightarrow$ $x+3=6+\cfrac{1}{3+\cfrac{1}{x+3}}$

$\Rightarrow$ $x-3=\dfrac{1}{\left(3+\dfrac{1}{x+3}\right)}\cdot\dfrac{x+3}{x+3}$ $\Rightarrow$ $x-3=\dfrac{x+3}{3x+10}$ $\Rightarrow$ $(3x+10)(x-3)=x+3$ $\Rightarrow$ $3x^2=33$

$\Rightarrow$ $x^2=11$ $\Rightarrow$ $x=\sqrt{11}$, taking the positive square root, since $x>0$.
Answer: c

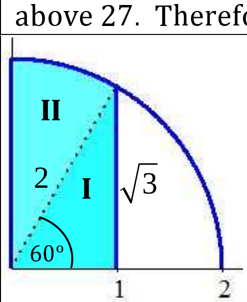
3.



Drawing the altitude *h*, as shown, gives two right triangles. Using the Pythagorean theorem for each, gives the equations: $x^2+h^2=\left(\sqrt{3}\right)^2$ and $(2-x)^2+h^2=\left(\sqrt{5}\right)^2$.
$\Rightarrow$ ❶ $x^2+h^2=3$ and ❷ $x^2-4x+h^2=1$ Solving ❶ for *x* gives $x=\sqrt{3-h^2}$, taking the positive square root. Substituting $x=\sqrt{3-h^2}$ into ❷ yields $3-h^2-4\sqrt{3-h^2}+h^2=1$ $\Rightarrow$ $4\sqrt{3-h^2}=2$ $\Rightarrow$ $3-h^2=\frac{1}{4}$ $\Rightarrow$ $h^2=\frac{11}{4}$ $\Rightarrow$ $h=\frac{\sqrt{11}}{2}$, again taking the positive square root. Since the area of a triangle is $\frac{1}{2}\,base\cdot height$, we get $\frac{1}{2}\cdot 2\cdot\frac{\sqrt{11}}{2}=\frac{\sqrt{11}}{2}$.
Answer: a

4. Clearly, any multiple of 5 can be purchased (5, 10, 15, ...). One box of 8 with two boxes 5 gives 18 nuggets, thus we can also order 28, 38 ... , by adding multiples of 10 (2 boxes of 5). One box of 8 and one box of 5 gives 13, hence we can order 23, 33, ... by adding multiples of 10. Two boxes of 8 gives 16 nuggets, thus we can get 26, 36, ... Continuing in this way (two boxes of 8 and one box of 5, three boxes of 8, three boxes of 8 and one of 5, four boxes of 8, four boxes of 8 and one of 5) we can get 21, 31, 41, ... ; 24, 34, 44, ... ; 29, 39, 49, ... ; 32, 42, 52, ... ; 37, 47, 57, ... Hence, any number of nuggets can be purchased above 27. Therefore, 27 is the maximum number that *cannot* be purchased.
Answer: 27

5.



Looking at the top half of the region, we can partition it into a 30°-60°-90° right triangle (Region I) with a base of 1 and height of $\sqrt{3}$, and a sector with an angle of 30° (Region II). The area of the triangle is $\frac{1}{2}\cdot 1\cdot\sqrt{3}=\frac{\sqrt{3}}{2}$, the area of the sector is $\frac{1}{12}$ the area of the entire circle (since 30° is $\frac{1}{12}$ of 360°) which is $\frac{1}{12}\pi\cdot 2^2=\frac{\pi}{3}$. Hence, the total area is $\frac{\sqrt{3}}{2}+\frac{\pi}{3}$. The entire shaded region is two times this value.
Answer: $\sqrt{3}+\dfrac{2\pi}{3}$

6\. One way to determine $P(x)$ is to replace x in $P(2x+3)$ with the inverse function of $2x+3$, which is $\frac{x-3}{2}$. Thus, $P(x)=P\left(2\left(\frac{x-3}{2}\right)+3\right)=8\left(\frac{x-3}{2}\right)^2+18\left(\frac{x-3}{2}\right)+10=2\left(x^2-6x+9\right)+9x-27+10$

$$=2x^2-12x+18+9x-17=2x^2-3x+1$$

Answer: $P(x)=2x^2-3x+1$

7\. Since $\sqrt[10]{2}$, $\sqrt[50]{213}$, $\sqrt[100]{2013}$ are all positive, raising each to the same positive power preserves their numerical ordering. Writing them as: $2^{\frac{1}{10}}$, $213^{\frac{1}{50}}$, $2013^{\frac{1}{100}}$, suggests raising each to the 100th power will give whole number results.

$$\Rightarrow \left(2^{\frac{1}{10}}\right)^{100}=2^{10}=1024,\ \left(213^{\frac{1}{50}}\right)^{100}=213^2>200^2=40000,\ \left(2013^{\frac{1}{100}}\right)^{100}=2013$$

$$\Rightarrow 2^{10}<2013<213^2 \Rightarrow 2^{\frac{1}{10}}<2013^{\frac{1}{100}}<213^{\frac{1}{50}}$$

Answer: c

8\. A trailing zero occurs for each factor of 10. Since 10=2·5, and there are far more factors of 2 in 2013! than factors of 5, we need only count the number of factors of 5. One factor of 5 occurs for every multiple of 5, an additional factor occurs for every multiple of 5^2 = 25, another factor for every multiple of 5^3 = 125, and yet one more factor for every multiple of 5^4 = 625 (5^5 = 3125 > 2013). Counting only the whole number of divisors, we get: 402 (from 2013÷5), 80 (from 2013÷25), 16 (from 2013÷125), and 3 (from 2013÷625). Thus, there are 402+80+16+3=501 factors of 5 (and 501 trailing zeros).

Answer: 501

9\. Let the three sides be represented by *a*, *b*, and *c*, with *a* = 15 and *c* being the hypotenuse. Thus, the Pythagorean theorem gives: $15^2+b^2=c^2 \Rightarrow (3\cdot5)^2=c^2-b^2 \Rightarrow (c-b)(c+b)=3\cdot3\cdot5\cdot5$, now we consider all possible two number factorings of $15^2=3\cdot3\cdot5\cdot5$. They are: $1\cdot225$, $3\cdot75$, $5\cdot45$, $9\cdot25$, and $15\cdot15$. Let's look at each of the five cases:

$(c-b)(c+b)=1\cdot225 \Rightarrow c-b=1$ and $c+b=225 \Rightarrow b=112, c=113$.

$(c-b)(c+b)=3\cdot75 \Rightarrow c-b=3$ and $c+b=75 \Rightarrow b=36, c=39$.

$(c-b)(c+b)=5\cdot45 \Rightarrow c-b=5$ and $c+b=45 \Rightarrow b=20, c=25$.

$(c-b)(c+b)=9\cdot25 \Rightarrow c-b=9$ and $c+b=25 \Rightarrow b=8, c=17$.

$(c-b)(c+b)=15\cdot15 \Rightarrow c-b=15$ and $c+b=15 \Rightarrow b=0, c=15$, not possible (a zero length)!

Thus, there are only four possible triangles.

Answer: c

10\. The probability we seek is $p_1\cdot p_2\cdot p_3\cdot p_4\cdot p_5\cdot p_6\cdot p_7$, where: p_n = the probability the n^{th} person was born on a different day than the $n-1$ people before him or her. Thus, $p_1=\frac{7}{7}$, $p_2=\frac{6}{7}$, $p_3=\frac{5}{7}$, $p_4=\frac{4}{7}$, $p_5=\frac{3}{7}$, $p_6=\frac{2}{7}$, and $p_7=\frac{1}{7}$. Therefore, the probability is $\frac{7}{7}\cdot\frac{6}{7}\cdot\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}\cdot\frac{2}{7}\cdot\frac{1}{7}=\frac{6}{7}\cdot\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}\cdot\frac{2}{7}\cdot\frac{1}{7}$.

Answer: b

11\. We know **I=0**, and **N=4**. We also know that **W=1**, since if the sum of two 4-digit numbers is a 5-digit number, it cannot exceed 9999+9999=19998. The 2nd column from the right has **M+E=E**, so either **M=0** or **M=9** and there is a 1 carry-over from the previous column (the 1st column). **M=9** and there is a 1 carry-over from the 1st column, since we already have **0** represented by **I**. In order for there to be a 1 carry-over from the 1st column, with **N=4**, **A** must be either a **6**, **7**, **8**, or **9**. A cannot be **6**, because then **R** would be **0**, but **I=0**; it cannot be **7**, because then **R** would be **1**, but **W=1**; it cannot be **9**, because **M=9**. Hence, **A=8** and **R=2**. The 4th column from the left has **B+I=N** or **B+0=4**. So, either **B=4** or **B=3** with a 1 carry-over from the 3rd column. However, **N=4**, so **B=3**. By similar deductions, **D=5** and **O=7**. Thus, **E** must be the only other digit not represented, which is **6**.

Answer: 6

12.	An odd function satisfies the following relation: $f(-x) = -f(x)$. $\arccos\left(\cos(-x)\right) = \arccos\left(\cos(x)\right)$, since cosine is an even function…thus $\arccos\left(\cos(x)\right)$ even (not odd). Answer: a <u>Note</u>: We can verify the other functions are odd! b) $\arcsin(-x) = -\arcsin(x)$, c) $\cos\left(\cos(-x)\right)\sin\left(\sin(-x)\right) = \cos\left(\cos(x)\right)\sin\left(-\sin(x)\right) = -\cos\left(\cos(x)\right)\sin\left(\sin(x)\right)$, and d) $\ln\left(\sqrt{(-x)^2+1}-(-x)\right) = \ln\left(\sqrt{x^2+1}+x\right) = -\ln\left(\frac{1}{\sqrt{x^2+1}+x}\right) = -\ln\left(\frac{1}{\sqrt{x^2+1}+x}\cdot\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}\right)$ $= -\ln\left(\frac{\sqrt{x^2+1}-x}{x^2+1-x^2}\right) = -\ln\left(\frac{\sqrt{x^2+1}-x}{1}\right) = -\ln\left(\sqrt{x^2+1}-x\right)$
13.	$\sum_{n=1}^{\infty}\sin^n(x) = \sin(x)+\sin^2(x)+\sin^3(x)+\sin^4(x)+\ldots = \frac{\sin(x)}{1-\sin(x)}$, using the formula for the sum of a geometric series, i.e. $\sum_{n=1}^{\infty} a r^n = \frac{a}{1-r}, -1<r<1$. Thus, $\frac{\sin(x)}{1-\sin(x)} = 2013 \Rightarrow \sin(x) = \frac{2013}{2014}$. Solving $\cos^2(x)+\sin^2(x)=1$ for $\cos(x)$, with $\sin(x) = \frac{2013}{2014}$, gives $\cos(x) = \sqrt{1-\left(\frac{2013}{2014}\right)^2}$, taking the positive root since x is an acute angle. $\cos(x) = \sqrt{\frac{2014^2}{2014^2}-\frac{2013^2}{2014^2}} = \sqrt{\frac{2014^2-2013^2}{2014^2}}$ $= \frac{\sqrt{(2014-2013)(2014+2013)}}{2014} = \frac{\sqrt{4027}}{2014}$ Answer: d
14.	Since no one is taking more than two languages, the only overlap (i.e. students in more than one class) is for students taking exactly two languages at a time. Summing the totals in each class, 150+175+200+225=750, counts all students who are taking two languages twice (e.g. 150+175 for French and Italian counts twice all those students who are taking both languages). Thus, we must subtract 100 from the 750, which gives 650 students in total. Answer: c
15.	Let's call the three people X, Y, and Z. Without loss of generality, we can assume X loses first, Y loses second, and Z loses last. Working backward: after the third game they each finished with \$32, so after the second game, X and Y each had \$16 each, thus Z had \$32+\$16+\$16=\$64. After the first game X had \$8 and Z had \$32, so Y had \$16+\$8+\$32=\$56. Therefore, at the start Z had \$16 and Y had \$28, so X had \$8+\$16+\$28=\$52. Answer: \$52
16.	$2\log(x-6y) = \log(x)+\log(y) \Rightarrow \log\left((x-6y)^2\right) = \log(xy) \Rightarrow (x-6y)^2 = xy$ $\Rightarrow x^2-12xy+36y^2 = xy$, dividing through by xy $\Rightarrow \frac{x}{y}-12+36\frac{y}{x}=1$ Letting $w=\frac{x}{y} \Rightarrow w-12+36w^{-1}=1$, multiplying by w and simplifying gives $w^2-13w+36=0 \Rightarrow (w-4)(w-9)=0 \Rightarrow w=\frac{x}{y}=4$ or 9, since x and y are positive. However, $x-6y>0 \Rightarrow \frac{x}{y}>6 \Rightarrow \frac{x}{y}=9$. Answer: 9

17.	$x=\sqrt{3}-\sqrt{2} \Rightarrow x^2=\left(\sqrt{3}-\sqrt{2}\right)^2 \Rightarrow x^2=3-2\sqrt{3}\sqrt{2}+2 \Rightarrow x^2-5=-2\sqrt{3}\sqrt{2}$, squaring again gives: $\left(x^2-5\right)^2=\left(-2\sqrt{3}\sqrt{2}\right)^2 \Rightarrow x^4-10x^2+25=4\cdot3\cdot2 \Rightarrow x^4-10x^2+1=0$ Answer: $P(x)=x^4-10x^2+1$ (or any non-zero integer multiple)
18.	Using $(x-h)^2+(y-k)^2=r^2$ for the equation of the circle, centered at (h,k) with radius *r*, and the three points give: ❶ $(0-h)^2+(0-k)^2=r^2 \Rightarrow h^2+k^2=r^2$, ❷ $(1-h)^2+(1-k)^2=r^2$, and ❸ $(11-h)^2+(1-k)^2=r^2$. Subtracting equation ❷ from equation ❸ gives $(11-h)^2-(1-h)^2=0$, whose only solution is $h=6$. Subtracting equation ❶ from equation ❷, with $h=6$ gives $-11+1-2k+k^2-k^2=0 \Rightarrow k=-5$. Thus, using equation ❶ with $h=6$ and $k=-5$ gives $36+25=r^2 \Rightarrow r=\sqrt{61}$. Answer: b
19.	Let n be the number of married couples at the party, so that there are $2n$ people. Each of the $2n$ people must shake hands with the other $2n-2$ people from all *other* couples. Thus, there are $2n(2n-2)$ total handshakes, but this counts each handshake twice (when A shakes B's hand, and when B shakes A's hand). Hence, we need to divide by 2, to obtain $2n(n-1)$. Now solve $2n(n-1)=264$. $2n^2-2n-264=0 \Rightarrow n^2-n-132=0 \Rightarrow (n-12)(n+11)=0 \Rightarrow n=12$ Answer: 12
20.	If the explorer brings one other man, who has 4 days of food supply, then after 1 day he heads back (using 1 day of food on the way and another day of food for the return trip). This leaves the explorer 2 additional days of food, but he can only carry 1 more day of food (having used 1 day of food himself). If the explorer brings two additional men, then after one of the helpers leaves after 1 day, the 2 remaining days of food (the helper needs a 1 day supply for the his return) can be carried, 1 for the explorer and 1 for the helper who remains. Thus, the explorer and his remaining helper both have a 4 day supply at the start of the 2nd day. At the end of the 2nd day, the remaining helper leaves, but he takes a 2 day supply with him for his return trip and gives the remaining 1 day supply to the explorer. Thus, at the end of the 2nd day, the explorer now has a 4 day supply – just enough to complete the 6-day trip. Answer: a

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