New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2014

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. The n^{th} iterate of function f is defined: $f^n(x) = \underbrace{f(f(f\cdots f}_{n}(x)))$, for example $f^3(x) = f(f(f(x)))$.

   If $f(x) = \dfrac{x-1}{x+1}$, then what is $f^{2014}(2014)$?

   a) $-\dfrac{2015}{2013}$ b) $-\dfrac{1}{2014}$ c) $\dfrac{2013}{2015}$ d) 2014

2. A man had collected several cars before deciding to sell some of them. During the first month, he sold half of them and half a car more. The second month, he sold half the remaining ones, plus half a car more. Finally, during the third month, again he sold half the remaining ones, plus half a car more. This left him with only one car. How many cars did he have before selling any?

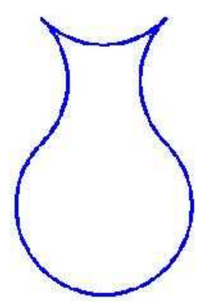
3. Two sides of a triangle have equal lengths of 10 inches. What is the maximum area this triangle can have (in square inches)?

   a) $25\sqrt{2}$ b) $25\sqrt{3}$ c) 50 d) $50\sqrt{2}$

4. Let w, x, y, and z be the four smallest consecutive positive integers such that w is a multiple of 3, x is a multiple of 5, y is a multiple of 7, and z is a multiple of 9. What is the value of w?

5. The diagram shows a shape composed entirely of arcs of radius 1, three ¼-arcs of a circle connected to a ¾-arc of a circle. What is the area enclosed by the figure?

   a) $\dfrac{5\pi}{4}$ b) 4 c) $\pi + \dfrac{3}{\pi}$ d) $\dfrac{4\pi}{3}$



6. If $f(x,y)=\dfrac{1}{x^2+4y^2-4xy+4}$, with x and y real numbers, what is the *maximum* value of f ?

7. If n is an integer greater than 21 and $21n$ is a perfect square, then what is the *minimum* value of n?

8. The *double factorial* of a positive integer is defined as: $n!!=\begin{cases} n\cdot(n-2)\cdot(n-4)\cdot\ldots\cdot 1, & n \text{ odd} \\ n\cdot(n-2)\cdot(n-4)\cdot\ldots\cdot 2, & n \text{ even}\end{cases}$.
For example: $5!!=5\cdot3\cdot1=15$ and $6!!=6\cdot4\cdot2=48$. Only *one* of the four choices below have the correct last few digits of $n!!$, where n is some integer between 50 and 60, inclusive. Which one?
a) ...89140825 b) ...76000000 c) ...77859275 d) ...08800000

9. I wish to meet with two of my friends at a location that is equidistant from all our homes. If I place an xy- coordinate system on the map, with my home at the origin of course, then my friend Sophia lives at the point $(10, 30)$ and Jay at $(20, 10)$, with coordinates in miles. How many miles must each of us travel?

10. If 2 marbles are removed at random from a bag containing blue and red marbles, the chance that they are both red is ⅓. If, instead, 3 are removed at random, the chance that they are all red is ⅙. How many marbles are blue?

11. If $a+b=1,\ b+c=2,\ c+d=3,\ d+e=4,\ e+f=5,\ \ldots\ ,\ y+z=25$ (i.e. the pattern continues for all the letters of the alphabet), then what is the average value of all variables (a through z)?

12. The graph of which relations consist of infinitely many parallel lines?
I. $\cos(x+y)+\sin(x+y)=1$ II. $2\cos(x+y)\sin(x+y)=1$ III. $\ln\left[2+\cos(x+y)\right]=e^{\sin(x+y)}$
a) I only b) II only c) I and II only d) I, II, and III

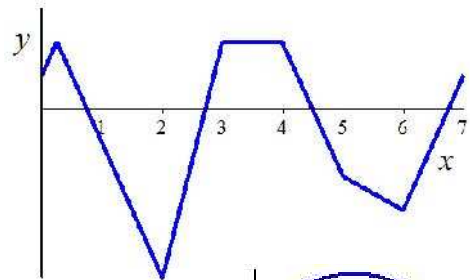
13. If $\cos(x)\sin(x)=\dfrac{1}{3}$ for $0°<x<90°$, what is $\cos(x)+\sin(x)$?
a) $\dfrac{\sqrt{3}}{2}$ b) $\dfrac{\sqrt{5}}{2}$ c) $\sqrt{\dfrac{3}{2}}$ d) $\sqrt{\dfrac{5}{3}}$

14. When bringing my daughter to college in Ithaca, we had a separate truck transport her luggage and other dorm items. Since the truck travels slower than us by car, we sent it on its way earlier. The truck averaged 45 miles/hour and arrived in Ithaca at 3:05 PM. We averaged 60 miles/hour and arrived at 2:47 PM (the same day). We both took the same route. How many miles before Ithaca did we pass the truck?

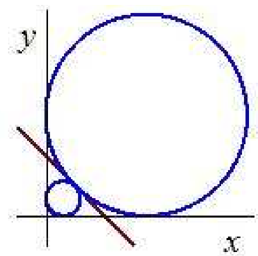
15. The number of hairs on a person's head ranges between approximately 90,000 and 150,000, with the average being about 120,000. The population of New York City is about 8 million people. Based only on this information, and excluding bald persons, the probability that there are two people in New York City that have *exactly* the same number of hairs on their head is
a) less than 0.10. b) greater than 0.10 and less than 0.50.
c) greater than 0.50 and less than 1. d) *exactly* 1.

16. How many real values of x satisfy the logarithmic equation $\log_x(2013-x)=2014$?
a) 0 b) 1 c) 2 d) More than 2.

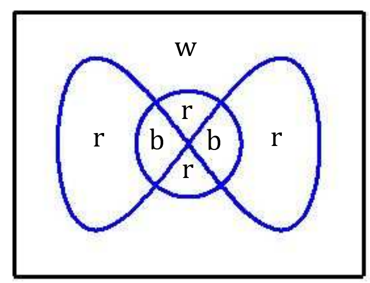
17. The graph of $y=f(x)$ is shown for $x\in[0,7]$.
If $f(x+7)=f(x)$, then how many solutions are there to $f(x)=0$ for $x\in[0,2014]$?



18. The diagram shows the two circles in the first quadrant that are tangent to the x-axis, y-axis, and the line $y=2-x$. What is the sum of the two radii?
a) $4\sqrt{2}-2$ b) 4 c) $\frac{1}{2}\left(1+5\sqrt{2}\right)$ d) $3\sqrt{2}$



19. The diagram shows the design for a new flag with one possible coloring. I want it colored so that no matter how it is oriented (upside-down or reversed) it will look the same. Furthermore, I do not want any two regions that share a border to have the same color. The colors red (r), white (w), and blue (b) are the only colors that may be used, and may use all three or only two of them. How many different flags are possible?



20. In a galaxy far, far away, there is a planet whose inhabitants have either 6, 7, or 8 legs. Those with 6 and 8 legs always tell the truth, while the 7-legged creatures always lie. One day, four of them gathered and one of them exclaimed "Altogether, we have 25 legs!," another stated "Altogether, we have 26 legs!," while another said "Altogether, we have 27 legs!," and finally the last one claimed "Altogether, we have 28 legs!" How many legs are there in total?
a) 25 b) 26 c) 27 d) 28

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Math League Contest ~ Spring 2014 ~ Solutions

1. $f(x)=\frac{x-1}{x+1} \Rightarrow f\left(f(x)\right)=f^{2}(x)=\frac{\left(\frac{x-1}{x+1}-1\right)}{\left(\frac{x-1}{x+1}+1\right)}=-\frac{1}{x} \Rightarrow f\left(f^{2}(x)\right)=f^{3}(x)=\frac{\left(-\frac{1}{x}-1\right)}{\left(-\frac{1}{x}+1\right)}=-\frac{x+1}{x-1}$

$\Rightarrow f\left(f^{3}(x)\right)=f^{4}(x)=\frac{\left(-\frac{x+1}{x-1}-1\right)}{\left(-\frac{x+1}{x-1}+1\right)}=x.$ Hence, every 4th iteration gives the original input. Since $\frac{2014}{4}$ is 503 with remainder 2, $f^{2014}(x)$ cycles 503 with 2 more iterations. Thus, $f^{2014}(2014)=f^{2}(2014)=-\frac{1}{2014}$.

Answer: b

2. Let x = the number of cars the man had at the start.

Month 1: He sold $\frac{1}{2}x+\frac{1}{2}$, leaving $x-\left(\frac{1}{2}x+\frac{1}{2}\right)=\frac{1}{2}x-\frac{1}{2}$.

Month 2: He sold $\frac{1}{2}\left(\frac{1}{2}x-\frac{1}{2}\right)+\frac{1}{2}=\frac{1}{4}x+\frac{1}{4}$, leaving $\left(\frac{1}{2}x-\frac{1}{2}\right)-\left(\frac{1}{4}x+\frac{1}{4}\right)=\frac{1}{4}x-\frac{3}{4}$.

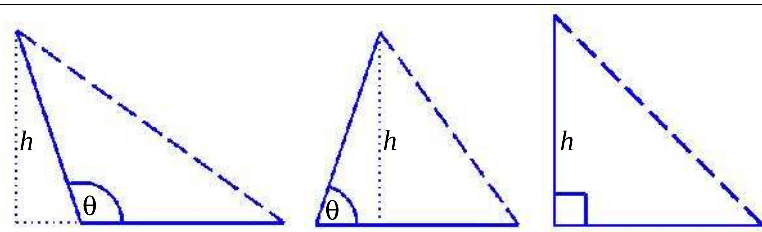
Month 3: He sold $\frac{1}{2}\left(\frac{1}{4}x-\frac{3}{4}\right)+\frac{1}{2}=\frac{1}{8}x+\frac{1}{8}$, leaving $\left(\frac{1}{4}x-\frac{3}{4}\right)-\left(\frac{1}{8}x+\frac{1}{8}\right)=\frac{1}{8}x-\frac{7}{8}$, which equals 1.

Therefore, $\frac{1}{8}x-\frac{7}{8}=1 \Rightarrow x=15$.

Answer: 15

Or, working backward…1 car remained, add $\frac{1}{2}$ then double to get 3. Add another $\frac{1}{2}$ and double to get 7. One more time, add $\frac{1}{2}$ then double to get 15.

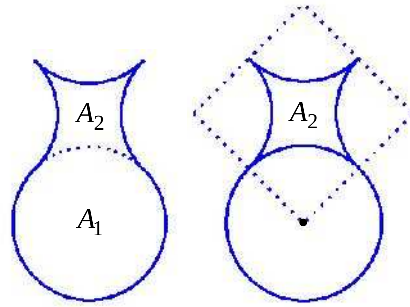
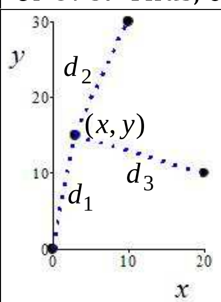
3. The base of each triangle has a length of 10, and 10 for the other leg (with the hypotenuse, dashed, and angle θ both variable). The height, h, is clearly maximized for $\theta=90^\circ$ (the last triangle). Hence, since the area is $\frac{1}{2}(base)(height)$, the maximum area is $\frac{1}{2}(10)(10)$ inches² = 50 inches².



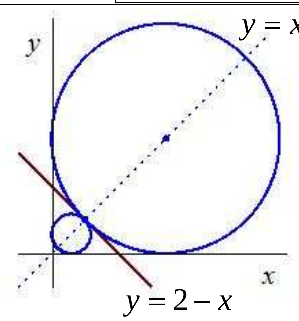
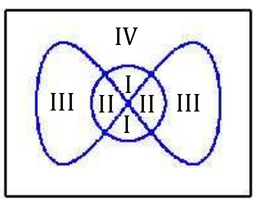
Answer: c

4. Since they are consecutive integers, $x=w+1$, $y=x+1=w+2$ and $z=x+2=w+3$. If z is a multiple of 9, then w must be a multiple of 3 (being $z-3$). Hence, we need only solve for x, y, and z being multiples of 5, 7, and 9, respectively. ① $x=5k$, ② $y=7m$, and ③ $z=9n$, where k, m, and n are positive integers. $z=x+2$ with ① and ③ gives $9n=5k+2 \Rightarrow n=\frac{5k+2}{9}$. $y=x+1$ with ① and ② gives $7m=5k+1 \Rightarrow m=\frac{5k+1}{7}$. The smallest positive integer for k which gives an integer for n is $k=5$, but that gives $m=\frac{26}{7}$, not an integer. In order to get the next k value for n, we increment it by 9 (since k is being divided by 9). Both 14 (5+9) and 23 (14+9) for k, yield non-integers for m. However, $k=32$ (23+9) gives $n=\frac{5\cdot 32+2}{9}=\frac{162}{9}=18$ and $m=\frac{5\cdot 32+1}{7}=\frac{161}{7}=13$. Thus, $x=5\cdot 32=160$, and $w=159$ (with $y=161$ and $z=162$).

Answer: 159

5.	The total area is $A_1 + A_2$, where $A_1 = \pi \cdot 1^2 = \pi$ (i.e. the area of the unit circle), while $A_2 = 2^2 - 4 \cdot \frac{1}{4}\pi \cdot 1^2 = 4 - \pi$ (i.e. the area of a square with edges of length 2, minus the area of the 4 quarter circles of radius 1). Hence, the area is $\pi + 4 - \pi = 4$. **Answer: b** 
6.	$f(x,y) = \frac{1}{x^2+4y^2-4xy+4}$ is *maximized* when the denominator is *minimized*. The denominator can be written as $x^2 - 4xy + 4y^2 + 4 = (x-2y)^2 + 4$, which is minimized when $(x-2y)^2 = 0$, i.e. when $x = 2y$. This gives $f_{\max} = \frac{1}{0+4} = \frac{1}{4}$. **Answer: $\frac{1}{4}$**
7.	$21n = 3 \cdot 7 \cdot n$ is a perfect square when its prime factors all have even exponents. $n = 3 \cdot 7$ works, giving $21n = 3^2 \cdot 7^2$, but $n > 21$. The next largest is $n = 2^2 \cdot 3 \cdot 7$, giving $21n = 2^2 \cdot 3^2 \cdot 7^2$. **Answer: $2^2 \cdot 3 \cdot 7 = 84$**
8.	If n is even, then $n!!$ will have a trailing "0" for each multiple of 5 encountered (since there are plenty of factors of 2 to give $2 \cdot 5$ as a factor). Thus, from 2 through 48 the only multiples of 5 are 10, 20, 30, and 40, which gives 4 trailing "0's." If $n = 50$, we get 2 more trailing "0's" (since 50 is a multiple of 5 twice, i.e. 5^2), thus giving a total of 6 trailing "0's" for any $n!!$ where n is even and between 50 and 58 (inclusive). Hence, choice "d" is eliminated. If n is odd and 25 or more, then $n!!$ will have at least 3 factors of 5, and will therefore be a multiple of 5^3=125. Since 1000 is a multiple of 125, the last three digits of any $n!!$ for n odd and 25 or more, will be: 125, 250, 375, 500, 625, 750, or 875. Thus, choices "a" and "c" are eliminated, leaving only "b" as viable. **Answer: b**
9.	 Let (x, y) be the coordinates of the meeting location. We require $d_1 = d_2$ and $d_2 = d_3$, which gives: $\sqrt{x^2 + y^2} = \sqrt{(x-10)^2 + (y-30)^2}$ and $\sqrt{(x-10)^2 + (y-30)^2} = \sqrt{(x-20)^2 + (y-10)^2}$. Solving these, by squaring both sides of each equation, yields $x = 5$ and $y = 15$. Thus, the distance is $\sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10}$ miles. **Answer: $\sqrt{250} = 5\sqrt{10}$**
10.	Let $r =$ the number of red marbles, and $n =$ the total number of marbles. The probability of selecting 2 red marbles is ① $\frac{r}{n} \cdot \frac{r-1}{n-1} = \frac{1}{3}$, while the probability of selecting 3 red marbles is ② $\frac{r}{n} \cdot \frac{r-1}{n-1} \cdot \frac{r-2}{n-2} = \frac{1}{6}$. Using ① in ② gives $\frac{1}{3} \cdot \frac{r-2}{n-2} = \frac{1}{6} \Rightarrow \frac{r-2}{n-2} = \frac{1}{2} \Rightarrow \frac{r-2}{n-2} = \frac{1}{2} \Rightarrow$ ③ $n = 2r - 2$. Substituting ③ into ① gives $\frac{r}{2r-2} \cdot \frac{r-1}{2r-3} = \frac{1}{3} \Rightarrow 3r(r-1) = (2r-2)(2r-3)$, which simplifies to $r^2 - 7r + 6 = 0 \Rightarrow (r-1)(r-6) = 0 \Rightarrow r = 1$ or $r = 6$. Only $r = 6$ makes sense, giving $n = 2 \cdot 6 - 2 = 10$. Thus, there are $10 - 6 = 4$ blue marbles. **Answer: 4**

11.	Adding all 25 equations gives ① $a+2b+2c+\cdots+2x+2y+z=1+2+3+\cdots+23+24+25$. The right-hand-side (RHS) of ① can quickly be determined by letting $A=1+2+3+\cdots+23+24+25$, which can also be written as $A=25+24+23+\cdots+3+2+1$. Adding these two, by adding the corresponding terms, gives $2A=26+26+26+\cdots+26+26+26$, the RHS consists of twenty-five 26's. Thus, $2A=25\cdot 26 \Rightarrow A=25\cdot 13$, making ① $a+2b+2c+\cdots+2x+2y+z=25\cdot 13$. Now if we take the alternating sum of the original 25 equations (i.e. we add, then subtract, add, subtract, ...), we get $(a+b)-(b+c)+(c+d)-\cdots+(w+x)-(x+y)+(y+z)=1-2+3-\cdots+23-24+25$. This reduces to $a+z=\underbrace{-1-1-\cdots-1}_{12}+25=-12+25$, and finally ② $a+z=13$. Adding ① and ② gives $2a+2b+2c+\cdots+2x+2y+2z=25\cdot 13+13=26\cdot 13.$, dividing by 2 gives the sum of all 26 variables: $a+b+c+\cdots+x+y+z=13\cdot 13$. Now divide by 26 to obtain the average, $\frac{a+b+c+\cdots+x+y+z}{26}=\frac{13\cdot 13}{26}=\frac{13}{2}$. Answer: $\frac{13}{2}=6.5$ Or, add every other equation: $(a+b)+(c+d)+(e+f)+\cdots+(w+x)+(y+z)=1+3+5+\cdots+23+25$ Which gives the sum of all 26 variables as 169 as well.
12.	Letting $w=x+y$, gives I. $\cos(w)+\sin(w)=1$, which is solved for $w=0$ and $w=\frac{\pi}{2}$, plus any integer multiple of 2π. Thus, $x+y=0+2\pi k$ and $x+y=\frac{\pi}{2}+2\pi k$, for k being any integer, giving infinitely many lines of slope -1. II. $2\cos(w)\sin(w)=1 \Rightarrow \cos(w)\sin(w)=\frac{1}{2}$, which has a solution $w=\frac{\pi}{4}$ (as well as $w=\frac{5\pi}{4}$), plus $2\pi k$. Similarly as above, this gives infinitely many lines of slope -1. III. $\ln\left(2+\cos(w)\right)=e^{\sin(w)}$, notice when $w=0$ the LHS > RHS (since $\ln(3)>1$), and when $w=\pi$ the LHS < RHS (since $0<e^{-1}$). Thus, there is a solution for between $w=0$ and $w=\pi$ (since the LHS and RHS are continuous functions), plus $2\pi k$. Now we reason as above to realize this also gives infinity many lines of slope -1. Answer: d
13.	Let $y=\cos(x)+\sin(x)$, then $y^2=\left[\cos(x)+\sin(x)\right]^2=\cos^2(x)+2\cos(x)\sin(x)+\sin^2(x)$. Now use $\cos^2(x)+\sin^2(x)=1$ and the given $\cos(x)\sin(x)=\frac{1}{3}$, we get $y^2=1+2\left(\frac{1}{3}\right)=\frac{5}{3} \Rightarrow y=\pm\sqrt{\frac{5}{3}}$. However, $0°<x<90°$, so we know $\cos(x)+\sin(x)>0$. Answer: d
14.	Let $x=$ the distance (in miles) between Ithaca and the point where the car passed the truck, and let $t=$ the time (in hours) it took the car to get to Ithaca once it passed the truck. Eighteen minutes is $\frac{18}{60}=\frac{3}{10}=0.3$ hour and since $distance=rate\cdot time$, we get ① $x=60t$, for the car, and ② $x=45(t+0.3)$, for the truck. Solving for x gives 54. Answer: 54
15.	It is *certain* that two people have the exact same number of hairs. If no two people have the same number, then each of the 8 million people must have a different number of hairs (requiring 8 million different values. However, there are *only* 60,001 (150,000 – 90,000 +1) different possible hair counts. Thus, there are *many* people in NYC with equal numbers of hairs on their head! Answer: d

16.	$\log_x(2013-x)=2014 \Rightarrow x^{2014}=2013-x$. The graph of $y=x^{2014}$ passes through the origin, is symmetric about the *y*-axis, and strictly increases as we move away from $x=0$ (it is "U" shaped). The graph of $y=2013-x$ is a line with a y-intercept of 2014 and slope of -1. Thus, the line must intersect the graph of $y=x^{2014}$ at two points, one for $x<0$ and one for $x>0$. However, the base of a logarithm must be greater than zero, giving only one real solution. Answer: b
17.	$f(x+7)=f(x)$ tells us f is periodic with a period of 7. $\frac{2014}{7}=287$ with a remainder of 5. Thus, for $0\le x\le 2014$ the graph cycles 287 times plus $\left(\frac{5}{7}\right)^{\text{th}}$ of a cycle. There are 4 roots in a full cycle, and 3 roots from $x=0$ to $x=5$. Therefore, there are $4\cdot 287+3=1151$ solutions. Answer: 1151
18.	By symmetry, we can see the two circles are tangent at $(1,1)$, i.e. where $y=2-x$ and $y=x$ intersect. The equation of the circles take the form $(x-r)^2+(y-r)^2=r^2$. They both contain the point $(1,1)$, giving $(1-r)^2+(1-r)^2=r^2$, which simplifies to $r^2-4r+2=0$. The quadratic formula gives $r=2\pm\sqrt{2}$, $r=2-\sqrt{2}$ for the smaller circle, and $r=2+\sqrt{2}$ for the larger circle. Hence, the sum is 4. Answer: b 
19.	 The regions are numbered (regions with the same numeral, must be colored the same) according to the symmetry required so the flag appears the same when reversed (left-right) or turned up-side down. Starting with Region IV, there are 3 possible color choices. Then, there are only 2 choices for Regions II. Now there are two different scenarios. ① Regions I and III are different colors. Thus, there is only 1 color choice for Regions I, also for Regions II (it must be the same as Region IV), giving $3\cdot 2\cdot 1\cdot 1=6$ different flags. ② Regions I and III are the same color. Now there are 2 color options for Regions II (same as IV or the yet unused color), giving $3\cdot 2\cdot 1\cdot 2=12$ different flags. Thus, there are $3\cdot 2\cdot 1\cdot 1=6+12=18$ possible flags. Answer: 18
20.	Suppose they are all liars, and hence 7-legged. Then there would be $4\cdot 7=28$ legs in total, making the last individual not a liar. Thus, they cannot all be liars. In that case, there must be *exactly one* truth-teller, since there cannot be two different correct number of legs in total. Therefore, there must be exactly 3 liars, each with 7 legs contributing 21 legs to the group. The truth-teller has either 6 or 8 legs. If (s)he had 8 legs, then there would be 21+8=29 legs, which was not offered as a total. Hence, the truth-teller must have 6 legs, giving a total of 21+6=27 legs. Answer: c

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