# New York State Mathematics Association of Two-Year Colleges

NYSMATYC 1967

## Math League Contest ~ Spring 2017

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. Let $f(x)$ be a function such that $(x-1)f(x)+xf\left(\frac{1}{x}\right)=x$, for all real numbers $x\neq 0$. What is the numerical value of $f(2)-f(1)$?

2. When I am as old as my father is now, I will be five times as old as my son is now. By then, my son will be 8 years older than I am now. The sum of my age and my father's age is 100. How old am I?

3. How many real values for *x* solve the equation $2016+2017^x=x^{2018}$?
   A) 1 B) 2 C) 3 D) more than 3

4. One day while looking at a clock on the wall, I noticed that the hour and minute hands overlap exactly every 65 minutes (by comparing it to a known accurate time-piece). By how many seconds, rounded to the nearest second, is the clock running too fast or too slow during the 65 minute period?
   A) 21 B) 24 C) 27
   D) None, the clock is accurate, the hands should overlap every 65 minutes.

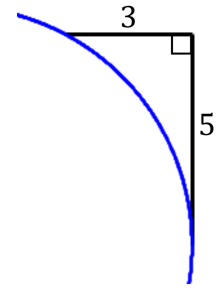
5. The NYSMATYC basketball team is a member of a six-team basketball league in which every pair of schools plays each other twice. The other five teams ended the season with league winning percentages of: 20%, 30%, 50%, 60% and 80%. What is the winning percentage record for the NYSMATYC team? Note: There are only wins and losses, no ties.

6. If the solutions to $ax^2+bx+c=0$, with $a\neq 0$ and $c\neq 0$, are $x_1=\alpha$ and $x_2=\beta$, then the solutions to $cx^2+bx+a=0$ are

   A) $x_1=\alpha$ and $x_2=\beta$ B) $x_1=\dfrac{1}{\alpha}$ and $x_2=\dfrac{1}{\beta}$

   C) $x_1=-\alpha$ and $x_2=-\beta$ D) $x_1=\alpha\beta$ and $x_2=\alpha+\beta$

7. Two perpendicular line segments of lengths 3 and 5 are drawn so they terminate on the circumference of a circle of radius $r$, as shown. If the segment of length 5 is tangent to the circle, then what is the radius, $r$, of the circle?




A) $\sqrt{34}$ B) $\frac{17}{3}$ C) $\frac{25}{4}$ D) $\frac{34}{5}$

8. At a gathering of 40 people, half of them know each other, while the other half know no one. The people that know each other hug, and those who do not know each other shake hands. How many handshakes occur? Assume that if person X knows person Y, then person Y knows person X; and if person X does not know person Y, then person Y does not know person X.

9. What real value for $x$ solves the equation $\log_x(x+1) = \log_{x+1}(x)$?

10. Which of the following functions has the greatest value for $x = 1.01$?

A) $f(x) = x$ B) $f(x) = \log_2(1+x)$ C) $f(x) = \log_{10}(1+9x)$ D) $f(x) = 0.01 + \sin\left(\frac{\pi}{2}x\right)$

11. The floor function, denoted $\lfloor x \rfloor$, is defined as the greatest integer less than or equal to $x$, e.g. $\lfloor 1.2 \rfloor = 1$, $\lfloor 1.9 \rfloor = 1$, and $\lfloor 2 \rfloor = 2$. How many positive integer solutions are there to the equation $\left\lfloor \frac{x}{8} \right\rfloor = \left\lfloor \frac{x}{10} \right\rfloor$?

12. Define $P(n)$ to be the product of all positive divisors of the integer $n$. For example, $P(18) = 1 \cdot 2 \cdot 3 \cdot 6 \cdot 9 \cdot 18 = 5832$. If $P(n) = 3^{15}$, then what is the value of $n$?

13. What is the coefficient of the $x^{100}$ term in the expansion of $\left(\sum_{n=0}^{100} x^n\right)^2$?

Note: $\sum_{n=0}^{100} x^n = 1 + x + x^2 + x^3 + \ldots + x^{97} + x^{98} + x^{99} + x^{100}$

14. The equation $x(x+1)(x+2)(x+3) = 3$ has two real and two complex solutions. What is the product of the two real solutions?

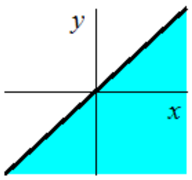
A) $-3$ B) $-1$ C) 1 D) 3

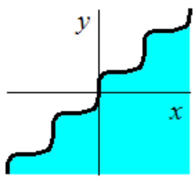
15. Suppose you agree to play the following game. You are given 3 blue marbles, 3 red marbles, and 2 bags. Your task is to distribute the marbles in the two bags in any fashion you desire. Another individual will randomly select one of the bags, then randomly select a marble from that bag. If a blue marble is drawn, you win \$100. If a red marble is drawn, you lose \$100. You obviously want to arrange the marbles so that your chances of winning are maximized. If you arrange the marbles in the optimal way in your favor, what is the probability you win?

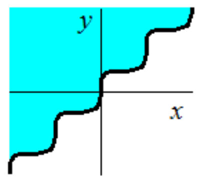
A) $\frac{3}{5}$ B) $\frac{2}{3}$ C) $\frac{7}{10}$

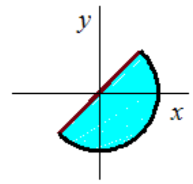
D) It does not matter how it is done, the probability of winning is $\frac{1}{2}$ in any case.

16. Which of the following shaded regions best represents the graph of $y-x \le \sin(y-x)$?

A) 

B) 

C) 

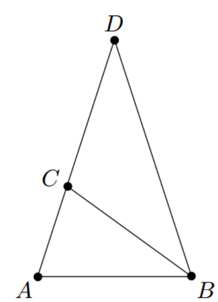
D) 


17. Letting $i=\sqrt{-1}$, the expression $\frac{(1+i)^{2017}}{(1-i)^{2015}}$ is equivalent to?

A) $-2i$ B) $-2$ C) $2i$ D) $2$

18. What is the *exact* value of $\arcsin\left(\sin(10)\right)$? Note: The angle, 10, is in radians.

A) $2\pi-10$ B) $3\pi-10$ C) $10-3\pi$ D) $10$




19. In the accompanying diagram, $\overline{AB}=\overline{BC}=\overline{CD}$ and $\overline{AD}=\overline{BD}$. What is the degree measure of angle $D$?

20. Five real numbers are represented by *a*, *b*, *c*, *d*, and *e*. We are given the following information about them:

i) $a=c$ if and only if $e \ne b$.

ii) Only if *c* is as much less than *b* as *b* is less than *a*, is *a* greater than *d*.

iii) $c<a$ and $c>d$.

Arrange these values in <u>descending</u> order (i.e. from largest to smallest). If two values are equal, then list in alphabetical order (for example, if $a=c$ then write it as "...*ac*..." rather than "...*ca*...").

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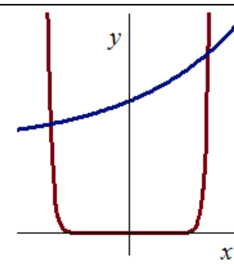
# New York State Mathematics Association of Two-Year Colleges

## Math League Contest ~ Spring 2017 ~ Solutions

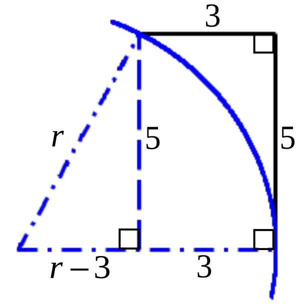
| | |
|---|---|
| 1. | First, letting $x=1$ gives: $(1-1)f(1)+1\cdot f(1)=1$, thus $f(1)=1$. Now let $x=2$ to give: $(2-1)f(2)+2\cdot f\left(\frac{1}{2}\right)=2$, or ① $f(2)+2f\left(\frac{1}{2}\right)=2$. Finally, let $x=\frac{1}{2}$ to give: $\left(\frac{1}{2}-1\right)f\left(\frac{1}{2}\right)+\frac{1}{2}\cdot f(2)=\frac{1}{2}$ or ② $-f\left(\frac{1}{2}\right)+f(2)=1$. Solving ① and ② yields: $f(2)=\frac{4}{3}$ (and $f\left(\frac{1}{2}\right)=\frac{1}{3}$). Therefore, $f(2)-f(1)=\frac{4}{3}-1=\frac{1}{3}$. Answer: $\frac{1}{3}$ |
| 2. | Let $x=$ my age now, $f=$ my father's age now, and $s=$ my son's age now. I will be as old as my father in $f-x$ years. Thus, the first two clues give: ① $f=5s$ and $s+(f-x)=x+8$ or ② $s+f-2x=8$. The third clue tells us ③ $x+f=100$. Solving equations ①, ②, and ③, yields $x=35$ (and $s=13$, $f=65$). Answer: 35 |
| 3. | The left-hand-side of the equation is an exponential growth function, while the right-hand-side is an even degree polynomial passing through the origin. Thus, there are clearly two points of intersection (i.e. two solutions to the equation), one near $x=-1$ and another near $x=1$, as shown. However, we know exponential growth will *always* out-pace any polynomial growth. Hence, the exponential function must again intersect the polynomial at some point for $x>>1$ and then stay greater than the polynomial. Therefore, there must be 3 points of intersection, for 3 solutions.   Answer: C (Note: The 3 solutions are: $x\approx-1.00378$, $x\approx1.00413$, and $x\approx2018.15134$.) |
| 4. | Since the minute hand makes a complete revolution every 60 minutes, it's angular speed is $\frac{360^\circ}{60\,\text{min}}=6^\circ/\text{min}=\frac{1}{10}{}^\circ/\text{sec}$. The hour hand makes a complete revolution every 12 hours, making it's angular speed $\frac{360^\circ}{720\,\text{min}}=\frac{1}{2}{}^\circ/\text{min}=\frac{1}{120}{}^\circ/\text{sec}$. Without loss of generality, let's examine the hands of a clock starting from the 12 o'clock position. One hour later the minute hand is back on the 12, while the hour hand is on the 1. Let $t=$ the time in seconds for the minute hand to next coincide with the hour hand from the 1 PM position. The minute hand is initially at the 0° position, and the hour hand starts displaced by 30°. Thus, we need to solve $0+\frac{1}{10}t=30+\frac{1}{120}t$ or equivalently $12t=3600+t$. This yields $t=\frac{3600}{11}$ seconds, or $t\approx327$ seconds (after 1 hour). 327 seconds is 5 minutes and 27 seconds. Thus, the hands of a clock should overlap about every 65 minutes 27 seconds. Hence, the clock in the problem is running too fast by about 27 seconds. Answer: C |
| 5. | Each team must play each of the other 5 teams twice, for a total of 10 games per team. The 20% winning team must then have 2 wins and 8 losses, the 30% winning team has 3 wins and 7 losses, the 50% winning team has 5 wins and 5 losses, the 60% winning team has 6 wins and 4 losses, while the 80% winning team must have 8 wins and 2 losses. Thus, the total number of wins for the 5 teams is $2+3+5+6+8=24$. However, there were a total of 30 games played $\left(2\cdot\frac{6\cdot5}{2}\right)$, and therefore 30 wins. That means the NYSMATYC team must have the other 6 wins, and 4 losses, for a 60% winning percentage. Answer: 60% |

6. Replacing $x$ with $\frac{1}{x}$ in $cx^2+bx+a=0$ gives: $c\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+a=0$, which is equivalent to $ax^2+bx+c=0$. This shows that if $x$ solves $ax^2+bx+c=0$, then $\frac{1}{x}$ solves $cx^2+bx+a=0$.

Answer: B

7. From the diagram, with the radii represented by the dash-dot lines, and the height of the right triangle formed represented by the dashed line, the Pythagorean theorem enables us to determine the radius. Specifically, $5^2+(r-3)^2=r^2 \Rightarrow 6r=34 \Rightarrow 6r=34 \Rightarrow r=\frac{17}{3}$.

Answer: B




<u>Alternate (Longer) Solution</u>

Calculate the area of the trapezoid formed by the two radii shown and the two lengths of 3 and 5. Also, determine the same area by summing the area of the right triangle formed by the lengths 3 and 5, and the triangle formed by the two radii and the hypotenuse $\sqrt{34}$. Then set the two areas, both in terms of *r*, equal and solve for *r*.

8. The 20 people who do not know each other shake each other's hands, for $\frac{20\cdot19}{2}=190$ handshakes. Then they each shake the hand of the 20 other people (who all know each other), for another $20\cdot20=400$ handshakes, for a total of $190+400=590$ handshakes.

Answer: 590

<u>Alternate Solution</u>

Among the 40 people there are total of $\frac{40\cdot39}{2}=780$ handshakes and hugs combined. Only the 20 people who know each other hug, which means there are $\frac{20\cdot19}{2}=190$ hugs. Thus, there must be $780-190=590$ handshakes.

9. Rewriting the equation as: $\frac{\log(x+1)}{\log(x)}=\frac{\log(x)}{\log(x+1)}$, gives $\log^2(x+1)=\log^2(x)$. Since $\log(x+1)\neq\log(x)$, we must have $\log(x+1)=-\log(x)$ or $x+1=\frac{1}{x}$. This gives us the quadratic equation: $x^2+x-1=0$, whose solutions are $x=\frac{-1\pm\sqrt{5}}{2}$. Since $x$ must be positive, we obtain only $x=\frac{-1+\sqrt{5}}{2}$.

Answer: $\frac{\sqrt{5}-1}{2}$

10. Choices A, B, and C are all increasing functions and have $f(1)=1$. Both logarithmic functions (B and C) are concave down, so they both have smaller values for $x>1$ than the linear function (A). Choice D has $f(1)=1.01$, which is its maximum value and then decreases. Thus, the function in D has $f(1.01)<1.01$. Hence, the linear function yields the largest value of $f(1.01)$.

Answer: A

11. For the positive integers $1\le x\le 7$, both sides of the equation give 0. For the integers $10\le x\le 15$, both sides of the equation give 1. For the integers $20\le x\le 23$, both sides of the equation give 2. For the integers 30 and 31, both sides of the equation give 3. For the integers $x\ge 32$, $\left\lfloor\frac{x}{8}\right\rfloor>\left\lfloor\frac{x}{10}\right\rfloor$.

Thus, there only 19 positive integer solutions:

$x\in\{1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 30, 31\}$.

Answer: 19

12. Since $P(n)$ is a power of the prime number 3, $n$ must also be a power of 3. Thus, $n=3^k$ and its factors are: $3^0$, $3^1$, $3^2$, $3^3$, … , $3^k$. Hence, $3^0\cdot3^1\cdot3^2\cdot\ldots\cdot3^{k-1}\cdot3^k=3^{0+1+2+\ldots+k}=3^{15}$, which means $1+2+\ldots+k=15$. Therefore, $k=5$ to give $n=3^5=243$.

Answer: 243

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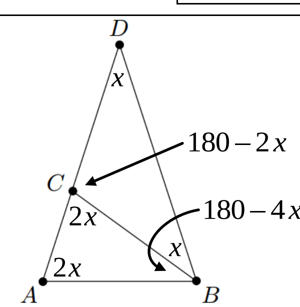
|---|---|
| 13. | $\left(1+x+x^2+x^3+\ldots+x^{97}+x^{98}+x^{99}+x^{100}\right)^2$ will yield an $x^{100}$ term whenever the sum of the exponents in the product is 100: $x^0\cdot x^{100}$, $x^1\cdot x^{99}$, $x^2\cdot x^{98}$, … , $x^{98}\cdot x^2$, $x^{99}\cdot x^1$, and $x^{100}\cdot x^0$. This gives 101 such terms, whose sum is $101x^{100}$. **Answer: 101** |
| 14. | Rewrite $x(x+1)(x+2)(x+3)=3$ as $[x(x+3)][(x+1)(x+2)]=3$, which is equivalent to $\left[x^2+3x\right]\left[x^2+3x+2\right]=3$. Now substitute $u=x^2+3x$, to give the equation $u(u+2)=3$ or $u^2+2u-3=0$, which factors to: $(u-1)(u+3)=0$. Thus, the original problem can now easily be expressed in factored form: $\left(x^2+3x-1\right)\left(x^2+3x+3\right)=0$. Only the first factor has real roots, whose product is the constant term $-1$. **Answer: B** |
| 15. | Each bag has a probability of $\frac{1}{2}$ of being selected, so we can maximize the total probability by placing a single blue marble in one bag (so if that bag is selected, the probability of a blue marble being drawn is 1). This leaves 5 marbles for the second bag, 2 blue and 3 red. Thus, if the second bag is selected, the probability a blue one is selected is $\frac{2}{5}$. Hence, the total probability of a blue marble being drawn is $\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{2}{5}=\frac{7}{10}$. **Answer: C** |
| 16. | First let's determine the graph of $y-x=\sin(y-x)$. Notice that if $y=x$ the equation is satisfied, so that $y=x$ satisfies the *equality*. For the *inequality*, selecting point not on the line $y=x$ shows the region below it satisfies the inequality. **Answer: A** |
| 17. | $\frac{(1+i)^{2017}}{(1-i)^{2015}}=\frac{(1+i)^{2015}(1+i)^2}{(1-i)^{2015}}=\left(\frac{1+i}{1-i}\right)^{2015}(1+i)^2=\left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^{2015}(1+2i-1)=\left(\frac{1+2i-1}{1+1}\right)^{2015}(2i)$ $=(i)^{2015}(2i)=(i)^3(2i)=(-i)(2i)=2.$ **Answer: D** |
| 18. | Since the angle $10-2\pi\approx 3.7$ radians, which is in the 3rd quadrant, then so is 10 radians, with a reference angle of $10-2\pi-\pi=10-3\pi$. Thus, $\arcsin(\sin(10))=\arcsin(-\sin(10-3\pi))$, since the sine of an angle in the 3rd quadrant is negative. Also, $\arcsin(-\sin(10-3\pi))=\arcsin(\sin(3\pi-10))$, since sine is an odd function. Hence, we can now write $\arcsin(\sin(3\pi-10))=3\pi-10$, since $-\frac{\pi}{2}\le\arcsin(x)\le\frac{\pi}{2}$ and certainly $3\pi-10$ is in that interval. **Answer: B** |
| 19. | Letting $x=\angle D$ (in degrees) and using the information given, deducing that triangles *ABC*, *ABD*, and *BCD* are isosceles, all the interior angles can then be labeled as shown. Since angles *ABD* and *BAC* are congruent, $180-4x+x=2x$ to give $x=36$. **Answer: 36°**   |
| 20. | Statement (iii) tells us that $c<a$, thus $a\ne c$ so that from (i), $e=b$. Statement (iii) also indicates that $c>d$, so we get the partial (descending) ordering: $a>c>d$. Now we only need to determine where $b$ and $e$ fit in. Statement (ii) can be rewritten as: *If* $a>d$, *then* $b-c=a-b$. Well, we now know that $a>d$. Thus, $a>b>c$ (in fact, $b=\frac{a+c}{2}$). Therefore, $a>b=e>c>d$. **Answer: $abecd$** |