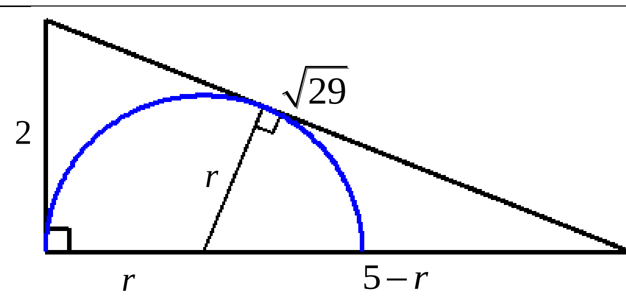
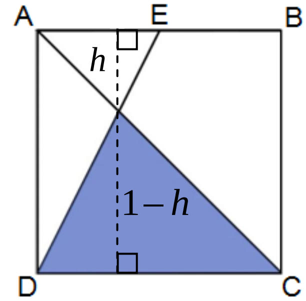
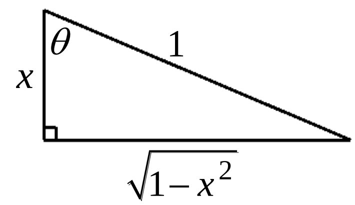
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2019 ~ Solutions

1.	Since $4\lfloor 4\rfloor\lceil 4\rceil = 64 < 99$ and $5\lfloor 5\rfloor\lceil 5\rceil = 125 > 99$, if there is a solution to $x\lfloor x\rfloor\lceil x\rceil = 99$, then $4 < x < 5$. Thus, $\lfloor x\rfloor = 4$ and $\lceil x\rceil = 5$, making $x\lfloor x\rfloor\lceil x\rceil = x\cdot 4\cdot 5 = 20x$. Therefore, $20x = 99$, giving $x = \frac{99}{20} = 4.95$. **Answer: $\frac{99}{20} = 4.95$**
2.	Let x = my age today, and y = my daughter's age today. Thus, ① $x+8=2(y+8)$ and ② $x-8=3(y-8)$. Solving these gives: $x=56$ and $y=24$. Hence, the sum is 80. **Answer: 80**
3.	$\frac{\sqrt{1-\sin^2(x)}}{\cos(x)}+\frac{\sqrt{1-\cos^2(x)}}{\sin(x)}=\frac{\sqrt{\cos^2(x)}}{\cos(x)}+\frac{\sqrt{\sin^2(x)}}{\sin(x)}=\frac{\lvert\cos(x)\rvert}{\cos(x)}+\frac{\lvert\sin(x)\rvert}{\sin(x)}=\pm1\pm1$ or $\pm1\mp1$. Thus, the sum could be $-2, 0,$ or 2. **Answer: E**
4.	Solving $\frac{1}{m}+\frac{1}{n}=\frac{2}{2019}$ for n gives: $n=\frac{2019m}{2m-2019}$. The smallest integer value for m that yields a positive integer for n is 1010, so that the denominator is 1 (any smaller integer value for m makes n negative). **Answer: 1010**
5.	$3!\,5!\,7! = 3\cdot2\cdot1\cdot5\cdot4\cdot3\cdot1\cdot7! = (2\cdot5)\cdot(3\cdot3)\cdot(2\cdot4)\cdot7! = 10\cdot9\cdot8\cdot7! = 10!$ **Answer: 10**
6.	Since $Area = \frac{1}{2}\cdot base\cdot height$, we can maximum the area by maximizing the base and height. If we allow one of the given sides to be the longest side, then the base or height would not be a maximum. Thus, neither side of length 4 or 5 can be the longest side. Letting the base be either 4 or 5, the height is maximized by making the base and height perpendicular. Thus, the Pythagorean theorem gives the third side (hypotenuse): $\sqrt{4^2+5^2}=\sqrt{41}$. **Answer: $\sqrt{41}$** <u>Alternate Solution</u>: $Area = \frac{1}{2}ab\sin(\theta)$, where θ is the angle between the two given sides, with $\sin(\theta)$ maximized when $\theta = 90°$. Now the third side can be obtained via the Pythagorean theorem as above.
7.	Let $x =$ the number of black socks, thus $10-x =$ the number of white socks. The probability of selecting 2 black socks in succession is then $\frac{x}{10}\cdot\frac{x-1}{9}$. Hence, $\frac{x}{10}\cdot\frac{x-1}{9}=\frac{1}{3} \Rightarrow x^2-x-30=0$ $\Rightarrow (x-6)(x+5)=0 \Rightarrow x=6,-5$. Therefore, there are 6 black socks, and 4 white socks. The probability of *not* getting a matching pair (i.e. one black and 1 white, in either order) is: $P(BW \text{ or } WB) = P(BW)+P(WB) = \frac{6}{10}\cdot\frac{4}{9}+\frac{4}{10}\cdot\frac{6}{9}=\frac{4}{15}+\frac{4}{15}=\frac{8}{15}$. **Answer: E**

8.	I. 100! has $\frac{100}{2^1}=50$ factors divisible by 2^1; $\frac{100}{2^2}=25$ factors divisible by 2^2; $\frac{100}{2^3}=12.5$, thus 12 factors divisible by 2^3, $\frac{100}{2^4}\approx 6.3$, thus 6 factors divisible by 2^4, $\frac{100}{2^5}\approx 3.1$, thus 3 factors divisible by 2^5, $\frac{100}{2^6}\approx 1.6$, thus 1 factor divisible by 2^6. Hence, 100! has $50+25+12+6+3+1=97$ 2's that factor it. Thus, 2^{97} is the greatest power of 2 that factors 100!, making $\frac{100!}{2^{98}}$ not a whole number. II. $\frac{\log_{10}\left(googol^{googol}\right)}{\log_{100}(googolplex)}=\frac{googol\cdot\log_{10}\left(10^{100}\right)}{\log_{100}\left(10^{googol}\right)}=\frac{googol\cdot 100}{googol\cdot\log_{100}(10)}=\frac{10^{100}\cdot 100}{10^{100}\cdot\left(\frac{1}{2}\right)}=200$ III. $\frac{\sqrt{1000^{googol}}}{googolplex}=\frac{\sqrt{100^{googol}10^{googol}}}{10^{googol}}=\frac{\sqrt{100^{googol}}\cdot\sqrt{10^{googol}}}{10^{googol}}=\frac{10^{googol}\cdot 10^{\frac{1}{2}\cdot googol}}{10^{googol}}=10^{\frac{1}{2}\cdot googol}$ $=10^{5\times 10^{99}}$ Clearly (now), II and III are whole numbers. **Answer: A**
9.	$w=\frac{1}{2}\cdot googol=\frac{1}{2}\cdot 10^{100}=5\times 10^{99}$, $x=\sqrt[100]{googolplex}=\left(10^{10^{100}}\right)^{\frac{1}{100}}=10^{10^{98}}$, $y=\ln(googolplex)=\ln\left(10^{googol}\right)=googol\cdot\ln(10)=\ln(10)\times 10^{100}\Rightarrow 2\times 10^{100}<y<3\times 10^{100}$, For $z=googol\cdot\cos\left(googol^\circ\right)$, the question is "approximately what is $\cos\left(googol^\circ\right)$ and how does it compare to $\frac{1}{2}$ (to compare z to w)?" We thus need to know the reference angle of $\left(10^{100}\right)^\circ$. $1000\div 360=2$ with a remainder of 280. Hence, multiplying 1000 by 10 would give 20 with a "remainder" of 2800...but $2800\div 360=7$ with a remainder of 280 as well. Therefore, repeated multiples of 10 will always leave a remainder of 280 when divided by 360. Hence, $\left(10^{100}\right)^\circ$ is coterminal to 280°, which is in the 4th quadrant with a reference angle of 80°. Thus, $\cos\left(googol^\circ\right)=\cos(80^\circ)$ which is $\cos\left(80^\circ\right)<\cos(60^\circ)=\frac{1}{2}$, making $z<w$. **Answer: z, w, y, x**
10.	Letting $r=$ the radius of the semicircle, and drawn to the hypotenuse of the triangle at the point of tangency, it forms a similar right triangle. By similar triangles, we obtain $\frac{r}{5-r}=\frac{2}{\sqrt{29}}\Rightarrow r=\frac{10}{2+\sqrt{29}}$ or $r=\frac{2}{5}\left(\sqrt{29}-2\right)$. **Answer: $\frac{10}{2+\sqrt{29}}=\frac{2}{5}\left(\sqrt{29}-2\right)$** 
11.	Let $f(x)=\alpha x+\beta$, where α and β are constants. I. Not true if f is a horizontal line (i.e. $\alpha=0$). II. True, since $f(ax+b)=\alpha(ax+b)+\beta=\alpha ax+\alpha b+\beta$, also a linear function. III. True if $\alpha=0$, making $f(\cos(x))=0\cdot\cos(x)+\beta=\beta$ **Answer: D**

12.	The two triangles formed are similar, as they have corresponding angles. Letting $h =$ the height of the smaller triangle, $1-h =$ the height of the larger triangle. By similar triangles we have: $\frac{h}{1/2} = \frac{1-h}{1} \Rightarrow 2h = 1-h \Rightarrow h = \frac{1}{3}$ Thus, the height of the larger triangle is $\frac{2}{3}$, making the area $\frac{1}{2}\cdot 1\cdot \frac{2}{3} = \frac{1}{3}$. Answer: C 
13.	For all triangles, the sum of the two smaller sides must be greater than the longest side. <u>Case I</u>: $\ln(x)$ is *not* the longest side, thus $\ln(4)$ must be. $\Rightarrow \ln(3)+\ln(x) > \ln(4) \Rightarrow \ln(3x) > \ln(4) \Rightarrow 3x > 4 \Rightarrow x > \frac{4}{3}$ <u>Case II</u>: $\ln(x)$ *is* the longest side. $\Rightarrow \ln(3)+\ln(4) > \ln(x) \Rightarrow \ln(12) > \ln(x) \Rightarrow 12 > x \Rightarrow x < 12$ Hence, $\frac{4}{3} < x < 12$, making the only integer solutions: 2 through 11 (10 integers). Answer: 10
14.	$\log_x(4) = \log_4(4x) \Rightarrow \frac{\log_4(4)}{\log_4(x)} = \log_4(4)+\log_4(x) \Rightarrow \frac{1}{\log_4(x)} = 1+\log_4(x)$ $\Rightarrow 1 = \log_4(x)+\log_4^2(x) \Rightarrow \log_4^2(x)+\log_4(x)-1=0$. Solving this via the quadratic formula gives: $\log_4(x) = \frac{1}{2}\left(-1\pm\sqrt{5}\right) \Rightarrow x_1 = 4^{\frac{1}{2}\left(-1-\sqrt{5}\right)} = 2^{-1-\sqrt{5}}$ and $x_2 = 4^{\frac{1}{2}\left(-1+\sqrt{5}\right)} = 2^{-1+\sqrt{5}}$. Hence, the product of the solutions is: $2^{-1-\sqrt{5}}\cdot 2^{-1+\sqrt{5}} = 2^{-2} = \frac{1}{4}$. Answer: C <u>Alternate Solution</u>: From $\log_4^2(x)+\log_4(x)-1=0$ (above), the sum of the roots is the negative coefficient of the $\log_4(x)$ term. Thus, $\log_4(x_1)+\log_4(x_2) = -1 \Rightarrow \log_4(x_1x_2) = -1 \Rightarrow x_1x_2 = 4^{-1}$.
15.	If the sine of the interior angles of a triangle are in the ratio $\sin(A):\sin(B):\sin(C) = 4:5:6$, then we also have $\frac{\sin(A)}{\sin(B)} = \frac{4}{5}$ and $\frac{\sin(B)}{\sin(C)} = \frac{5}{6}$. The Law of Sines then gives: $\frac{\sin(A)}{4} = \frac{\sin(B)}{5} = \frac{\sin(C)}{6}$, which means we can construct a triangle with sides $a=4$, $b=5$, and $c=6$. Now we use the Law of Cosines to obtain: $4^2 = 5^2+6^2-2\cdot5\cdot6\cos(A) \Rightarrow \cos(A) = \frac{-45}{-60} = \frac{3}{4}$ $5^2 = 4^2+6^2-2\cdot4\cdot6\cos(B) \Rightarrow \cos(B) = \frac{-27}{-48} = \frac{9}{16}$ $6^2 = 4^2+5^2-2\cdot4\cdot5\cos(C) \Rightarrow \cos(C) = \frac{-5}{-40} = \frac{1}{8}$ Thus, $\cos(A):\cos(B):\cos(C) = \frac{3}{4}:\frac{9}{16}:\frac{1}{8} = 12:9:2$. Answer: C

16.	Letting $\theta = \arccos(x)$ gives $\cos(\theta) = x$ and the accompanying right triangle. Also, $\theta = \arctan(x)$, which gives $\tan(\theta) = x$. From the triangle, we get $\tan(\theta) = \frac{\sqrt{1-x^2}}{x}$. Thus, $\frac{\sqrt{1-x^2}}{x} = x \Rightarrow x^2 = \sqrt{1-x^2}$, squaring both sides gives: $\frac{\sqrt{1-x^2}}{x} = x \Rightarrow x^4 = 1 - x^2 \Rightarrow x^4 + x^2 - 1 = 0$. The quadratic formula gives $x^2 = \frac{-1 \pm \sqrt{5}}{2}$. For this problem $x^2 > 0$, so we take $x^2 = \frac{-1+\sqrt{5}}{2} \Rightarrow x = \sqrt{\frac{\sqrt{5}-1}{2}}$ (taking the positive root since $\arccos(x) = \arctan(x)$ can only be true for $x > 0$).  Answer: D
17.	The $4xy + y^2$ term suggests we can write part of the left hand side as a perfect square in terms of x and y. $13x^2 = 4x^2 + 9x^2$, so the equation can be written: $\left(4x^2 + 4xy + y^2\right) + \left(9x^2 - 6x + 1\right) = 0$. Which can be expressed as: $(2x+y)^2 + (3x-1)^2 = 0$. Hence, $(2x+y)^2 = 0$ and $(3x-1)^2 = 0$. This gives: $y = -2x$ and $x = \frac{1}{3} \Rightarrow x = \frac{1}{3}$ and $y = -\frac{2}{3}$. Answer: $\left(\frac{1}{3}, -\frac{2}{3}\right)$
18.	Letting $y = x - 4$ transforms the equation to: $(y+3)(y+1)(y-1)(y-3) = 20$. Grouping conjugate pairs: $(y+3)(y-3)(y+1)(y-1) = 20 \Rightarrow \left(y^2-9\right)\left(y^2-1\right) = 20 \Rightarrow y^4 - 10y^2 - 11 = 0$. This last expression factors: $\left(y^2-11\right)\left(y^2+1\right) = 0 \Rightarrow y = \pm\sqrt{11}$, ignoring the imaginary solutions. Hence, $x - 4 = \pm\sqrt{11} \Rightarrow x = 4 \pm \sqrt{11}$ with $x = 4 + \sqrt{11}$ being the larger real solution. Answer: $4+\sqrt{11}$
19.	$\sin(k\pi x) = \frac{1}{2}$ when $k\pi x = \frac{\pi}{6} + 2\pi n$ or $k\pi x = \frac{5\pi}{6} + 2\pi n$, $n = 0, 1, 2, \ldots$ (we want $n \geq 0$ so $k > 0$). Solving for x gives: $x_1 = \frac{1+12n}{6k}$ or $x_2 = \frac{5+12n}{6k}$. Since we seek solutions at $x = 4$, we get: $k_1 = \frac{1+12n}{24}$ and $k_2 = \frac{5+12n}{24}$. When $n = 0$, $k_1 = \frac{1}{24}$ yields one solution; when $n = 0$, $k_1 = \frac{5}{24}$ yields two solutions; when $n = 1$, $k_1 = \frac{13}{24}$ yields three solutions; when $n = 1$, $k_2 = \frac{17}{24}$ yields four solutions; when $n = 1$, $k_2 = \frac{25}{24}$ yields five solutions. Hence, $\frac{17}{24} \leq k < \frac{25}{24}$. Answer: $k \in \left[\frac{17}{24}, \frac{25}{24}\right)$ <u>Note</u>: The interval *includes* $k = \frac{17}{24}$ so that the 4th solution occurs at $x = 4$, while we must *exclude* $k = \frac{25}{24}$ so the 5th solution "just misses" being included at $x = 4$.
20.	Letting $B =$ the number of boys in the family, and $G =$ the number of girls in the family... Each girl has $G-1$ sisters and B brothers, giving ① $B = G - 1$. Each boy has G sisters and $B-1$ brothers, giving ② $G = 2(B-1)$. Solving ① and ② gives $B = 3$ and $G = 4$. Answer: 7