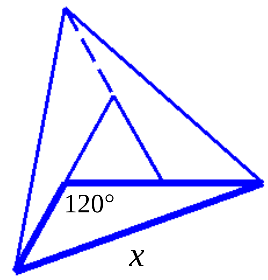
New York State Mathematics Association of Two-Year Colleges

NYSMATYC ∞ 1967

Math League Contest ~ Fall 2022 ~ Solutions

1.	$f(x)=\frac{x}{2^x-1}+\frac{x}{2}=\frac{x(2^x+1)}{2(2^x-1)} \Rightarrow f(-x)=\frac{-x(2^{-x}+1)}{2(2^{-x}-1)}=\frac{-x(1+2^x)}{2(1-2^x)}=\frac{x(2^x+1)}{2(2^x-1)}=f(x)$, so f is an even function. $g(x)=\frac{x}{2^x-1}-\frac{x}{2}=\frac{x(3-2^x)}{2(2^x-1)} \Rightarrow g(-x)=\frac{-x(3-2^{-x})}{2(2^{-x}-1)}=\frac{x(3\cdot 2^x-1)}{2(2^x-1)}\neq g(x)$. Thus, g is not an even function. Since g is not even, $f\cdot g$ is not even. Answer: A
2.	$\dfrac{\dfrac{\dfrac{\frac{1}{1+1}}{1+1+1}}{1+1+1+1}}{\dfrac{1+1+1+1}{\dfrac{1+1+1}{\frac{1+1}{1}}}}=\dfrac{\dfrac{\dfrac{\left(\frac{1}{2}\right)}{1+1+1}}{1+1+1+1}}{\dfrac{1+1+1+1}{\dfrac{1+1+1}{\left(\frac{2}{1}\right)}}}=\dfrac{\dfrac{\dfrac{\left(\frac{1}{2}\right)}{3}}{1+1+1+1}}{\dfrac{1+1+1+1}{\left(\frac{1+1+1}{2}\right)}}=\dfrac{\dfrac{\left(\frac{1}{6}\right)}{4}}{\dfrac{4}{\left(\frac{3}{2}\right)}}=\dfrac{\left(\frac{1}{24}\right)}{\left(\frac{8}{3}\right)}=\frac{1}{24}\cdot\frac{3}{8}=\frac{1}{64}$ Answer: B
3.	In order for three Tuesdays of a month to fall on even-numbered days, the first Tuesday of the month must land on an even-numbered day (otherwise only two Tuesdays could fall on even-numbered days). Certainly the first Tuesday of the month can fall on the 2nd, so that the subsequent Tuesdays would land on the 9th, 16th, 23rd, and 30th – giving three Tuesdays falling on even-numbered days. Can the first Tuesday fall on the 4th of the month? If so, then the subsequent Tuesdays would have to land on the 11th, 18th, 25th, and then the 32nd. Since no month has more than 31 days, the only way three Tuesdays can fall on even-numbered days is as first described. Hence, the 28th day of the month must be a Sunday. Answer: C
4.	Rather than compare the fractions by obtaining a common denominator, which would be quite cumbersome to do by hand, we can obtain a common numerator (a much easier task to perform by hand) and then compare denominators. The *least common numerator* is $3\cdot5\cdot7=105$. Thus, $x=\frac{3}{3002002}\cdot\frac{5\cdot7}{5\cdot7}=\frac{3\cdot35}{3002002\cdot35}=\frac{105}{105070070}$, $y=\frac{5}{5003009}\cdot\frac{3\cdot7}{3\cdot7}=\frac{5\cdot21}{5003009\cdot21}=\frac{105}{105063189}$, and $z=\frac{7}{7005001}\cdot\frac{3\cdot5}{3\cdot5}=\frac{7\cdot15}{7005001\cdot15}=\frac{105}{105075015}$. The smallest is the one with the largest denominator, followed by the one with the second largest denominator, then the one with the smallest denominator. Thus, from smallest to largest we have: z, x, y. Answer: z, x, y
5.	Clearly, $C=1$ since the sum of two 1-digit numbers and a 2-digit number cannot exceed $9+9+99=117$. B must be either 8 or 9, because if B were 7 or less, the sum of BB and two 1-digit numbers would yield only a 2-digit number. If $B=8$, then $A+A=23$ so that $A+A+88=111$. However, $A+A$ cannot be odd. Hence, $B=9$ with $A=6$. Answer: 6

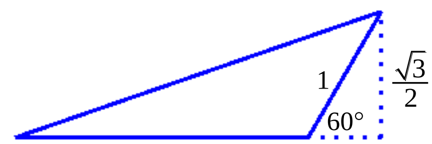
6. The area of an equilateral triangle with sides of length s is $\frac{\sqrt{3}}{4}s^2$. Thus, the area of the original (smaller) triangle is $\frac{\sqrt{3}}{4}$. Let x be the length of the sides of the new (larger) triangle. Analyzing one of the three new obtuse triangles formed (in bold), we can determine x. Using the Law of Cosines, we get: $x^2 = 1^2 + 2^2 - 2\cdot 1\cdot 2\cos(120°) \Rightarrow x^2 = 5 - 4(-\cos(60°)) = 5 - 4\left(-\frac{1}{2}\right)$. Thus, $x^2 = 5 + 2 = 7$, making the area of the new (larger) triangle $\frac{\sqrt{3}}{4}\cdot 7$. Hence, the ratio of the areas is: $\frac{\frac{\sqrt{3}}{4}\cdot 7}{\frac{\sqrt{3}}{4}} = 7$.



Answer: 7

Alternate Solution

Take one of the three obtuse triangles that surround the original triangle, using the side length of 2 as the base, and extend the base so we can form the height (both shown as dotted lines). The height of this triangle is the same as the height of the original equilateral triangle, but this triangle has a base of 2 (twice that of the original). Thus, the area is 2 times the area of the original triangle. The new larger triangle consists of three of these obtuse triangles plus the original one. Hence, the new larger triangle has an area that is $3\cdot 2 + 1 = 7$ times the original.



7. Across

1. If a number is both a perfect square and a perfect cube, then it must be a perfect 6th power. The only 3-digit perfect 6th power is 729 (i.e. $3^6 = 27^2 = 9^3 = 729$). This answer is a permutation of the digits 7, 2, and 9.
2. The only 3-digit perfect 4th powers are: $4^4 = 256$ and $5^4 = 625$ ($3^4 = 81$ and $6^4 = 1296$). Thus, this answer is either 256 or 625.
3. The 3-digit perfect cubes are: $5^3 = 125$, $6^3 = 216$, $7^3 = 343$, $8^3 = 512$, and $9^3 = 729$.

Down

1. Not helpful.
2. Helpful later.
3. From 2 across, we see the only choices are 256 and 625. However, it cannot be 625, since that would make 2 across have a "2" in the units digit. Thus, 3 down must be 256, making 2 across 625. This gives a "6" in the units digit of 3 across, making it 216. From 1 acorss, we know only the digits "7" and "9" remain. However, the reverse of 2 down is prime, so it cannot be 921, since it's reverse is not prime ($129 = 3\cdot 43$). Hence, 2 down must be 721 (with 127 being prime), forcing a "9" in the only remaining position.

1 9	2 7	3 2
2 6	2	5
3 2	1	6

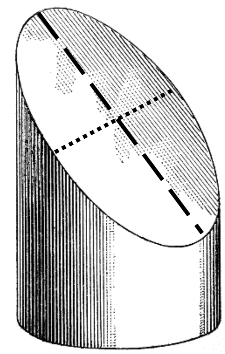
Answer: 216

8. From above, we get 962.

Answer: 962

9. Let n be the number of colors in the artist's palette. There are 2^n possible mixtures, since there are 2 choices for each color (it may be used or not used). However, this includes not using any color (1 way) and using only one color (n ways). The closest power of 2 to 500 is $2^9 = 512$. Subtracting the 1 way of using no colors and the 9 ways of using only one color, gives 502 mixtures, which slightly exceeds 500. Hence, 8 colors must be the maximum.

Answer: 8

10.	Only perfect squares have an odd number of factors, since non-perfect squares have unique pairs of factors whose product yields the number. Thus, we seek perfect squares between 2000 and 2100. Observing that $44^2 < 2000$, $45^2 = 2025$, and $46^2 > 2100$, 2025 is the only perfect square between 2000 and 2100. Thus, 2025 is the only number between 2000 and 2100 that has an odd number of factors (15 of them). Answer: 2025
11.	 The dotted line corresponds to the minor axis of the ellipse, which has a length equal to the diameter of the cylinder, making it 1. The dashed line corresponds to the major axis of the ellipse. The major axis is the hypotenuse of a right triangle with a base of 1 (the diameter of the cylinder) and a 60° angle at the base. Thus, the hypotenuse, i.e. major axis, has a length of $\frac{1}{\cos(60°)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$. An ellipse centered at the origin has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; here $a = \frac{2}{2} = 1$, and $b = \frac{1}{2}$. Thus, the equation becomes: $\frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1 \Rightarrow x^2 + 4y^2 = 1$. Answer: A
12.	$\frac{2022}{\log\left(4x^2 - 4x + 101\right)}$ is maximized when the denominator is the smallest positive value it can achieve. The quadratic $4x^2 - 4x + 101$ is minimized for $x = -\frac{-4}{2 \cdot 4} = \frac{1}{2}$ (i.e. on the axis of symmetry), giving the minimum $4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 101 = 1 - 2 + 101 = 100$. Thus, the denominator has a minimum value of $\log(100) = 2$. Therefore, $f_{\max} = \frac{2022}{2} = 1011$. Answer: B
13.	Since the maximum values of the cosine and sine functions are 1, the only way $\cos(3A - B) + \sin(A + 3B) = 2$ is for $\cos(3A - B) = 1$ and $\sin(A + 3B) = 1$. Thus, ① $3A - B = 0°$ and ② $A + 3B = 90°$. Solving ① and ②, yields $A = 9°$ and $B = 27°$. Hence, angle C must be $180° - (9° + 27°) = 144°$. Answer: 144°
14.	For any point (x, y) in the plane, the distance from the origin is $\sqrt{x^2 + y^2}$. Thus, maximizing $x^2 + y^2$ is equivalent to maximizing the distance (x, y) is from the origin. Hence, the problem can be restated: *What is the maximum distance between the origin and a point on the circle given by* $(x + 2)^2 + (y - 4)^2 = 45$? The points on a circle that are closest and furthest from the origin lie on the line through the origin and the center of the circle. The line through $(0, 0)$ and $(-2, 4)$ is $y = -2x$. This line and the circle intersect where: $(x + 2)^2 + (-2x - 4)^2 = 45 \Rightarrow 5(x + 2)^2 = 45$ $\Rightarrow (x + 2)^2 = 9 \Rightarrow x + 2 = \pm 3 \Rightarrow x = -5$ or $x = 1$. This gives the points on the circle at extreme distances from the origin: $(-5, 10)$ and $(1, -2)$, with $(-5, 10)$ being the furthest. Therefore, the maximum value of $x^2 + y^2$ is $(-5)^2 + 10^2 = 125$. Answer: 125 <u>Alternate (Intuitive) Solution</u> The center of the circle, $(-2, 4)$, is $\sqrt{(-2)^2 + 4^2} = \sqrt{20}$ units from the origin. The radius of the circle is $\sqrt{45}$. Thus, the maximum distance (discussed above) is $\sqrt{20} + \sqrt{45} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$. Hence, the maximum value of $x^2 + y^2$ is $\left(5\sqrt{5}\right)^2 = 125$.

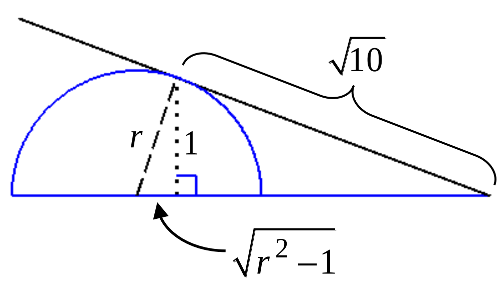
15. Since $\log_y(x)=\dfrac{\log_b(x)}{\log_b(y)}=\left(\dfrac{\log_b(y)}{\log_b(x)}\right)^{-1}=\left(\log_x(y)\right)^{-1}=\frac{1}{\log_x(y)}$, the equation can be written as $\log_x(y)+2022=2023\frac{1}{\log_x(y)}$. Multiplying by $\log_x(y)$ and rearranging gives: $\left(\log_x(y)\right)^2+2022\log_x(y)-2023=0$. The last equation factors, giving: $\left(\log_x(y)-1\right)\left(\log_x(y)+2023\right)=0$. Thus, $\log_x(y)-1=0 \Rightarrow y=x$ or $\log_x(y)+2023=0 \Rightarrow y=x^{-2023}$, making the product $x\cdot x^{-2023}=x^{-2022}$. Answer: -2022

16. $x^2+bx+r=0$ will have real solutions only when $b^2-4\cdot1\cdot r\geq 0$, from the quadratic formula. Thus, we need to determine the probability that $b^2\geq 4r$. Since there are 6 outcomes for each roll of each die, there are a total of $6\cdot6=36$ possible outcomes for the combined roll. Now we merely need to count how many of those 36 rolls will have $b^2\geq 4r$. When $r=1$, b could be 2, 3, 4, 5, or 6; when $r=2$, b could be 3, 4, 5, or 6; when $r=3$, b could be 4, 5, or 6; when $r=4$, b could be 4, 5, or 6; when $r=5$, b could be 5, or 6; and when $r=6$, b could be 5, or 6. This gives 19 ways out of the 36 possible rolls (with each outcome equally likely). Thus, the probability is $\frac{19}{36}$. Answer: D

17. If the divisors were unlimited, then we should choose 2023 or larger, so the remainder would be 2022. However, 1000 is our maximum divisor. Thus, we must select a divisor that divides 2022 *almost* a whole number of times. Half of 2022 is 1011, so we would like to divide by 1012 to obtain a remainder of 1010, but 1012 is too big (over 1000). The next best choice would be a divisor that divides 2022 almost 3 times. Since $2022\div3=674$, we should divide by 675, which yields a remainder of 672 as the maximum. Answer: 672

18. $y^3-y^2+y+x^2y-x^2=1\Rightarrow y^3-y^2+y-1+x^2y-x^2=0\Rightarrow\left(y^3-y^2\right)+\left(y-1\right)+\left(x^2y-x^2\right)=0$
Factoring by grouping: $y^2(y-1)+(y-1)+x^2(y-1)=0 \Rightarrow (y-1)\left(y^2+1+x^2\right)=0$, and since $y^2+1+x^2\geq1$, the last equation can only be satisfied when $y-1=0$, i.e. when $y=1$. Answer: B

19. After drawing the radius (the dashed line) to the point of tangency, we see that the new (small) right triangle with the radius as the hypotenuse is similar to the larger right triangle with hypotenuse $\sqrt{10}$. The similar triangles relationship gives: $\frac{\sqrt{r^2-1}}{r}=\frac{1}{\sqrt{10}} \Rightarrow \frac{r^2-1}{r^2}=\frac{1}{10} \Rightarrow r^2=\frac{10}{9} \Rightarrow r=\frac{\sqrt{10}}{3}$.



Answer: $\frac{\sqrt{10}}{3}$

20. Statements A, B, C, and E can all be said by the person whether lying or not. If the person is telling the truth, then statement D would be a lie; if the person is lying, then statement D would be true. Hence, statement D can never be said by an inhabitant of Logic Land. Answer: D