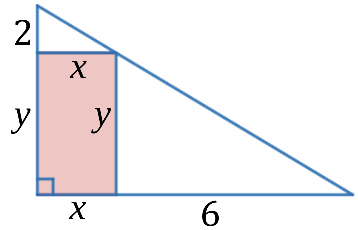
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2023 ~ Solutions

1.	For the numerator: let $x=\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{\cdots}}}}}}$, giving $x^2=6\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{\cdots}}}}}=6x$. Thus, $x^2-6x=0 \Rightarrow x(x-6)=0 \Rightarrow x=0$ or $x=6$. Since x is clearly greater than 0, $x=6$. For the denominator: let $y=\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}}$, giving $y^2=6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}=y+6$. Thus, $y^2-y-6=0 \Rightarrow (y+2)(y-3)=0 \Rightarrow y=-2$ or $y=3$. Since y is clearly greater than 0, $y=3$. Hence, the limiting value is $\frac{x}{y}=\frac{6}{3}=2$. Answer: D
2.	Let $R=\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$. Thus, $R=\frac{x}{a-b}$, $R=\frac{y}{b-c}$, and $R=\frac{z}{c-a}$. These give: $x=R(a-b)=aR-bR$, $y=R(b-c)=bR-cR$, and $z=R(c-a)=cR-aR$. Therefore, $x+y+z=aR-bR+bR-cR+cR-aR=0$. Answer: B
3.	The vendor has incurred a loss of \$100, the amount of the counterfeit money. The customer no longer has the phony \$100 bill, nor does the other street merchant. Another way to see it is that the vendor kept the real \$25 change, but gave \$100 back to the street merchant, so the vendor has lost \$75 there. Additionally, the vendor lost the \$25 value of the item taken by the customer. Answer: D
4.	$\underbrace{123\ldots9}_{9\text{ digits}}\underbrace{101112\ldots99}_{2\cdot 90=180\text{ digits}}\underbrace{100101102\ldots999}_{3\cdot 900=2700\text{ digits}}1000\ldots2023$ Thus, the 2023rd digit falls among the 3-digit number sequence, it is the $2023-(9+180)=1834^{\text{th}}$ digit after the "99." $1834\div 3=611$ with a remainder of 1. Hence, the digit we seek is the first digit after the first 611 3-digit numbers (after "99"). The 611th 3-digit number is $611+99=710$. The next digit after that is the "7" of 711. Answer: 7
5.	For $1^3+2^3+3^3+4^3+\ldots+2022^3+2023^3$, we need only keep track of the units digits. Let "$\rightarrow$" mean "only the units digit." Thus, the sum becomes: $\rightarrow \underbrace{1+8+7+4+5+6+3+2+9+0}_{10\text{ digit cycle}} \quad + \underbrace{\quad . \quad . \quad . \quad}_{219\text{ more cycles}} + \underbrace{1+8+7}_{\text{from } 2021^3+2022^3+2023^3}$ $=\underbrace{45+45+\ldots+45}_{220}+16 \rightarrow 0+6=6$. Hence, the units digit is "6." Answer: 6
6.	Letting m and n be the two numbers: $GCF(m,n)=9=3^2$ and $LCM(m,n)=270=2\cdot 3^3\cdot 5$. Thus, one of the numbers, m, must have 3^2 as a factor and the other, n, must have a factor of 3^3. Now we just need to "distribute" the 2 and 5. Here are the possibilities: ① $m=3^2\cdot 2\cdot 5$ and $n=3^3$, or ② $m=3^2\cdot 2$ and $n=3^3\cdot 5$, or ③ $m=3^2\cdot 5$ and $n=3^3\cdot 2$, or ④ $m=3^2$ and $n=3^3\cdot 2\cdot 5$. Thus, there are four different pairs. Answer: 4

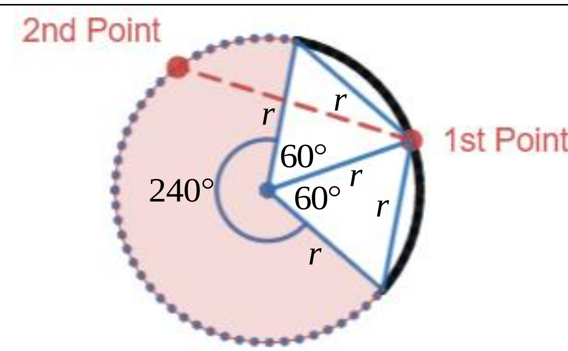
7. Let the base of the rectangle be x, and the height be y, as shown. The two triangles are similar, since they have corresponding angles. Thus, $\frac{2}{x}=\frac{y}{6} \Rightarrow xy=12$. Hence, the area is 12. $\boxed{\text{Answer: C}}$



8. $4^y=27 \Rightarrow (2^2)^y=3^3 \Rightarrow 3=2^{\frac{2}{3}y}$ Substituting this into the other equation gives: $(2^{\frac{2}{3}y})^x=32 \Rightarrow 2^{\frac{2}{3}xy}=2^5 \Rightarrow \frac{2}{3}xy=5 \Rightarrow xy=\frac{15}{2}$. $\boxed{\text{Answer: D}}$

9. The integers between $11!+1$ and $11!+12$ are: $11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1+n$, where $n=2,3,4,5,6,7,8,9,10,11$. However, it is clear that for each of those n-values, n is a factor. Therefore, none of those integers can be prime. $\boxed{\text{Answer: A}}$

10. The first point chosen can be anywhere on the circumference (shown as "1st Point" in the diagram). In order for the distance between the first and second points (the dashed line) to be greater than the radius, r, the second point must fall on the dotted arc that bounds the shaded region, which has an angle of $240°$. Thus, the probability the second point randomly falls on that arc is $\frac{240°}{360°}=\frac{2}{3}$. $\boxed{\text{Answer: } \frac{2}{3}}$



11. Using the change of base relation $\log_b(x)=\frac{\log_6(x)}{\log_6(b)}$, the expression can be written as:

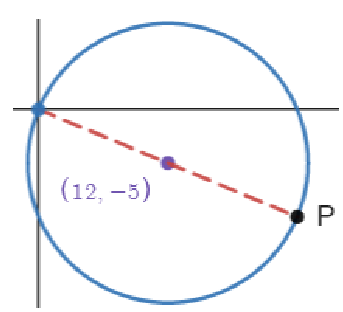
$$\frac{1}{\left(\frac{\log_6(6)}{\log_6(9)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(12)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(32)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(81)}\right)}=\frac{\log_6(9)}{\log_6(6)}+\frac{\log_6(12)}{\log_6(6)}+\frac{\log_6(32)}{\log_6(6)}+\frac{\log_6(81)}{\log_6(6)}$$

$$=\frac{\log_6(3^2)}{1}+\frac{\log_6(2^2\cdot3)}{1}+\frac{\log_6(2^5)}{1}+\frac{\log_6(3^4)}{1}=\log_6(3^2\cdot2^2\cdot3\cdot2^5\cdot3^4)=\log_6(2^7\cdot3^7)$$

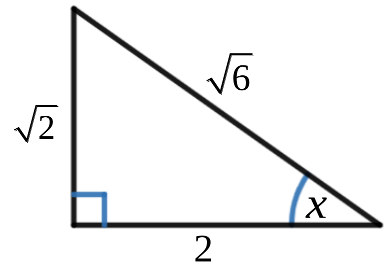
$$=\log_6(6^7)=7.$$

$\boxed{\text{Answer: C}}$

12. $x^2+y^2=\left(\sqrt{x^2+y^2}\right)^2$, i.e. it is the square of the distance from $(0,0)$ to (x,y), where the point (x,y) is on the graph of $x^2+y^2=24x-10y$. Completing the square for x's and y's gives: $(x-12)^2+(y+5)^2=169=13^2$. Thus, the graph is a circle centered at $(12,-5)$ with radius 13. The point on the circle furthest from the origin will give the maximum value of x^2+y^2. The endpoints of the diameter on the line through origin are the points closest and furthest away from the origin. Since the circle passes through the origin, the furthest point is the other endpoint of the diameter, point P. The distance equals the diameter of the circle, which is 26. Hence, $x^2+y^2=26^2=676$. $\boxed{\text{Answer: D}}$



13. Letting $x = \arctan\left(\frac{\sqrt{2}}{2}\right)$ gives the right triangle shown, since the tangent of an angle in a right triangle is $\frac{\textit{opposite}}{\textit{adjacent}}$. Using the double angle formula for sine, $\sin(2x) = 2\cos(x)\sin(x)$, and the triangle to get $\cos(x) = \frac{2}{\sqrt{6}}$ and $\sin(x) = \frac{\sqrt{2}}{\sqrt{6}}$, gives $\sin(2x) = 2 \cdot \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{2}}{\sqrt{6}}$. Which simplifies to $\frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$.



$\boxed{\text{Answer: D}}$

14. $x + y = 4 \Rightarrow (x+y)^3 = 4^3 \Rightarrow x^3 + 3x^2y + 3xy^2 + y^3 = 64 \Rightarrow x^3 + y^3 + 3xy(x+y) = 64$ $\Rightarrow 8 + 3xy \cdot 4 = 64 \Rightarrow 12xy = 56 \Rightarrow xy = \frac{14}{3} \Rightarrow \frac{x+y}{xy} = \frac{4}{\left(\frac{14}{3}\right)} \Rightarrow \frac{1}{y} + \frac{1}{x} = \frac{12}{14} = \frac{6}{7}$.

$\boxed{\text{Answer: } \frac{6}{7}}$

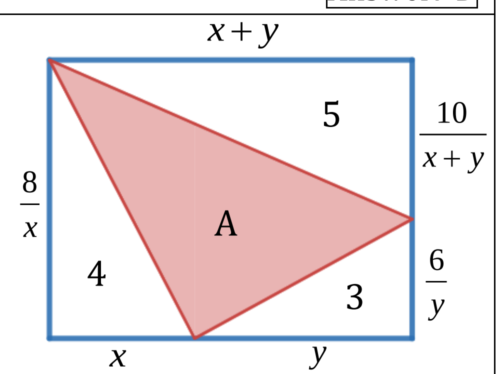
15. Taking $\log_2$ of both sides of $x^{2+\log_2(x)} = 2$ gives: $\left(2 + \log_2(x)\right)\log_2(x) = 1$. Now letting $y = \log_2(x)$ gives: $(2+y)y = 1 \Rightarrow y^2 + 2y - 1 = 0 \Rightarrow y = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$. Thus, $\log_2(x_1) = -1 - \sqrt{2}$ and $\log_2(x_2) = -1 + \sqrt{2} \Rightarrow x_1 = 2^{-1-\sqrt{2}}$ and $x_2 = 2^{-1+\sqrt{2}}$, making $\frac{x_2}{x_1} = \frac{2^{-1+\sqrt{2}}}{2^{-1-\sqrt{2}}} = 2^{2\sqrt{2}} = 4^{\sqrt{2}}$.

$\boxed{\text{Answer: } 2^{2\sqrt{2}} = 4^{\sqrt{2}}}$

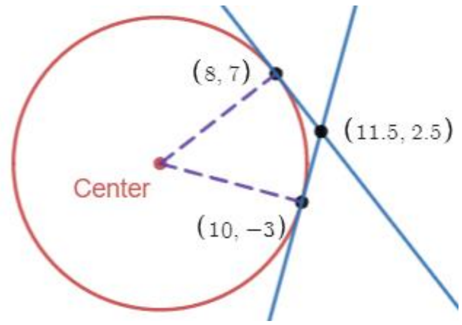
16. Since the period of $\cos(nx)$ is $\frac{2\pi}{n}$, the period of $\cos(x)$ is 2π, the period of $\cos(2x)$ is π, the period of $\cos(3x)$ is $\frac{2\pi}{3}$, and so on through the period of $\cos(2023x)$ being $\frac{2\pi}{2023}$. Hence, the period of the sum of all those periodic functions is the largest of the periods, i.e. the period of $\cos(x)$ which is 2π – since there is no smaller value of $P > 0$ for which $\cos(x+P) = \cos(x)$.

$\boxed{\text{Answer: D}}$

17. Let the base of the triangle with an area of 4 be x, making its height $\frac{8}{x}$; and the base of the triangle with an area of 3 be y, making its height $\frac{6}{y}$. Thus, the width of the triangle with an area of 5 is $x+y$, and its height is $\frac{10}{x+y}$. This gives: $\frac{10}{x+y} + \frac{6}{y} = \frac{8}{x}$. Multiplying the last equation by $xy(x+y)$ and simplifying results in: $3x^2 + 4xy - 4y^2 = 0$, which factors as $(3x-2y)(x+2y) = 0$. Solving for y gives: $y = \frac{3}{2}x$ or $y = -\frac{1}{2}x$, but both x and y must be positive, making $y = \frac{3}{2}x$. Hence, the area of the rectangle is $\frac{8}{x}(x+y) = \frac{8}{x}(x + \frac{3}{2}x) = \frac{8}{x} \cdot \frac{5}{2}x = 20$. We can finally obtain the area, A, of the shaded triangle: $A = 20 - (3+4+5) = 8$.



$\boxed{\text{Answer: 8}}$

18.	Since $\cos(x)=\sin(x+\frac{\pi}{2})$ and they both have a period of 2π, then from $\sin(y)=\cos(x)$ we get $y=x+\frac{\pi}{2}+2\pi k$, for $k=0, \pm 1, \pm 2, \pm 3, \ldots$. These graphs are a family of parallel lines with slopes of 1. Also, we know $\cos(x)=\sin(\frac{\pi}{2}-x)$, so $\sin(y)=\cos(x)$ becomes $\sin(y)=\sin(\frac{\pi}{2}-x)$ together with the periodicity we finally obtain $y=\frac{\pi}{2}-x+2\pi k$, for $k=0, \pm 1, \pm 2, \pm 3, \ldots$. These graphs are a family of parallel lines with slopes of -1. Checking symmetry: replacing x with $-x$ gives: $\sin(y)=\cos(-x) \Leftrightarrow \sin(y)=\cos(x)$ – symmetric about the y-axis; replacing y with $-y$ gives: $\sin(-y)=\cos(x) \Leftrightarrow -\sin(y)=\cos(x) \not\Leftrightarrow \sin(y)=\cos(x)$ – not symmetric about the x-axis. Hence, graph (A). $\boxed{\text{Answer: A}}$
19.	The slope of the tangent line at $(8,7)$ is $\frac{7-2.5}{8-11.5}=\frac{4.5}{-3.5}=-\frac{9}{7}$, and the slope of the tangent line at $(10,-3)$ is $\frac{2.5-(-3.5)}{11.5-10}=\frac{5.5}{1.5}=\frac{11}{3}$. Since tangent lines to a circle are perpendicular to the radius at the point of tangency, the slope of the radius from the center to $(8,7)$ is $\frac{7}{9}$, and the slope of the radius from the center to $(10,-3)$ is $-\frac{3}{11}$.  Thus, the equation of the line that contains the radius from the center to $(8,7)$ is $y-7=\frac{7}{9}(x-8) \Rightarrow y=\frac{7}{9}x+\frac{7}{9}$, and the equation of the line that contains the radius from the center to $(10,-3)$ is $y-(-3)=-\frac{3}{11}(x-10) \Rightarrow y=-\frac{3}{11}x-\frac{3}{11}$. The center is at the intersection of those two lines, which is at $(-1,0)$. Hence, the radius is the distance between $(-1,0)$ and either of the two points of tangency. Therefore, $r=\sqrt{(8-(-1))^2+(7-0)^2}=\sqrt{81+49}=\sqrt{130}$. $\boxed{\text{Answer: } \sqrt{130}}$
20.	Card 3 must be turned over to make sure there is a vowel on the other side. Card 4 must be checked as well, to make sure there is not an even number on the other side. The other cards need not be turned over, since no matter what is on the other side of them, the statement would not be violated. $\boxed{\text{Answer: C}}$

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