

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2019

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π , etc.

1. The *floor function*, denoted $\lfloor x \rfloor$, is defined as *the greatest integer less than or equal to x* ; while the *ceiling function*, denoted $\lceil x \rceil$, is defined as *the smallest integer greater than or equal to x* . For example, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.8 \rfloor = -3$, and $\lfloor -2 \rfloor = -2$; with $\lceil 2.1 \rceil = 3$, $\lceil -2.1 \rceil = -2$, and $\lceil -2 \rceil = -2$. What real number, if any, satisfies the equation: $x \lfloor x \rfloor \lceil x \rceil = 99$?
2. In eight years, I will be twice as old as my daughter. Eight years ago, I was three times as old as my daughter. What is the sum of our ages today?
3. For $\cos(x) \neq 0$ and $\sin(x) \neq 0$, the expression $\frac{\sqrt{1-\sin^2(x)}}{\cos(x)} + \frac{\sqrt{1-\cos^2(x)}}{\sin(x)}$ could equal which of the following values?
A) -2 only B) 0 only C) 2 only D) -2 or 2 only E) $-2, 0, \text{ or } 2$
4. For positive integers m and n , $\frac{1}{m} + \frac{1}{n} = \frac{2}{2019}$. What is the smallest possible value for m ?
5. Some integer, n , satisfies the equation $3!5!7! = n!$. What is n ?
6. Two sides of a triangle have length 4 and 5. What should be the length of the third side in order to maximize the area of the triangle?

7. I have 10 socks in a drawer, some black and some white. If 2 socks are selected at random, the probability that they are both black is $\frac{1}{3}$. What is the probability the 2 socks selected are *not* a matching pair? Assume each sock has the same probability of being selected.

- A) $\frac{2}{9}$ B) $\frac{4}{15}$ C) $\frac{1}{3}$ D) $\frac{4}{9}$ E) $\frac{8}{15}$

8. Which of the following is/are *not* a whole number? Note: $n! \equiv n(n-1)(n-2)\cdots 2 \cdot 1$, $googol \equiv 10^{100}$ and $googolplex \equiv 10^{googol}$.

- I. $\frac{100!}{2^{98}}$ II. $\frac{\log_{10}(googol^{googol})}{\log_{100}(googolplex)}$ III. $\frac{\sqrt{1000^{googol}}}{googolplex}$

- A) I only B) II only C) III only D) I and II only E) II and III only

9. Place the following numbers in numerical order, from smallest to largest (left to right): $w = \frac{1}{2} \cdot googol$, $x = \sqrt[100]{googolplex}$, $y = \ln(googolplex)$, $z = googol \cdot \cos(googol^\circ)$. Thus, if you believe $w < x < y < z$, for example, then your answer should be: w, x, y, z . See the previous problem for the definitions of a *googol* and a *googolplex*.

10. A semicircle is inscribed in a right triangle, with legs of length 2 and 5, so it is tangent to both the hypotenuse and the vertical leg of the triangle, as shown. What is the radius of the semicircle?



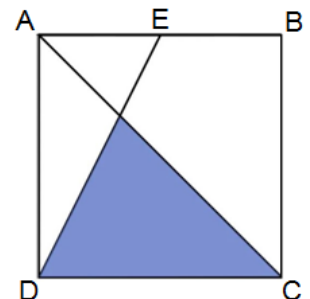
11. If $f(x)$ is a linear function, then which of the following *must* be true?

- I. $f^{-1}(x)$ is also a linear function.
 II. $f(ax+b)$ is also a linear function for all real numbers a and b .
 III. $f(\cos(x))$ *may* also be a linear function.

- A) I only B) II only C) I and II only D) II and III only E) I, II, and III

12. The figure shows a square with an area of 1. Segment AC is a diagonal, and DE extends from a corner of the square to the midpoint of AB. What is the area of the shaded region?

- A) $\frac{3}{10}$ B) $\frac{5}{16}$ C) $\frac{1}{3}$ D) $\frac{3}{8}$ E) $\frac{2}{5}$

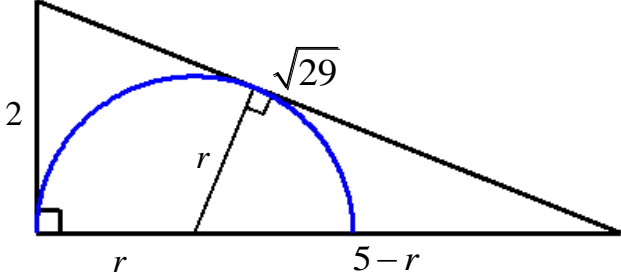


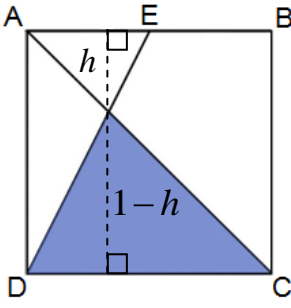
13. Three sides of a triangle have lengths $\ln(3)$, $\ln(4)$, and $\ln(x)$. How many different integer values are possible for x ?

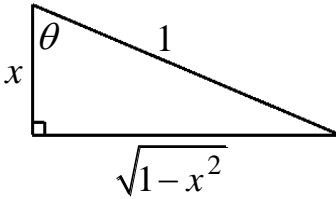
14. What is the product of all solutions of the equation $\log_x(4) = \log_4(4x)$?
- A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) 1
15. If a triangle has interior angles A , B , and C such that $\sin(A) : \sin(B) : \sin(C) = 4 : 5 : 6$, then $\cos(A) : \cos(B) : \cos(C) = ?$
- A) 4:5:6 B) 6:5:4 C) 12:9:2 D) 15:12:10 E) $\sqrt{84} : \sqrt{75} : 8$
16. If $\arccos(x) = \arctan(x)$, then x is
- A) $\frac{\pi}{5}$ B) $\sqrt{\sqrt{5}-\sqrt{3}}$ C) $\sqrt{3}-1$ D) $\sqrt{\frac{\sqrt{5}-1}{2}}$ E) $\frac{1}{2}\sqrt{1+\sqrt{3}}$
17. The graph of $13x^2 + 4xy + y^2 - 6x + 1 = 0$ consists of a single point in the xy -plane. What are the coordinates of that point?
18. The equation $(x-1)(x-3)(x-5)(x-7) = 20$ has two distinct real solutions (and two complex solutions). What is the larger of the two real solutions?
19. What positive values of k make the equation $\sin(k\pi x) = \frac{1}{2}$ have exactly four solutions on $x \in [0, 4]$? Express your answer in interval form.
20. A certain family has several children, some girls and some boys. Each girl has just as many sisters as brothers. Each boy has twice as many sisters as brothers. How many children are in this family?

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1.	<p>Since $4\lfloor 4\rfloor\lceil 4\rceil = 64 < 99$ and $5\lfloor 5\rfloor\lceil 5\rceil = 125 > 99$, if there is a solution to $x\lfloor x\rfloor\lceil x\rceil = 99$, then $4 < x < 5$. Thus, $\lfloor x\rfloor = 4$ and $\lceil x\rceil = 5$, making $x\lfloor x\rfloor\lceil x\rceil = x \cdot 4 \cdot 5 = 20x$. Therefore, $20x = 99$, giving $x = \frac{99}{20} = 4.95$.</p> <div style="float: right; border: 1px solid black; padding: 2px;">Answer: $\frac{99}{20} = 4.95$</div>
2.	<p>Let $x =$ my age today, and $y =$ my daughter's age today. Thus, ① $x + 8 = 2(y + 8)$ and ② $x - 8 = 3(y - 8)$. Solving these gives: $x = 56$ and $y = 24$. Hence, the sum is 80. Answer: 80</p>
3.	$\frac{\sqrt{1 - \sin^2(x)}}{\cos(x)} + \frac{\sqrt{1 - \cos^2(x)}}{\sin(x)} = \frac{\sqrt{\cos^2(x)}}{\cos(x)} + \frac{\sqrt{\sin^2(x)}}{\sin(x)} = \frac{ \cos(x) }{\cos(x)} + \frac{ \sin(x) }{\sin(x)} = \pm 1 \pm 1 \quad \text{or} \quad \pm 1 \mp 1.$ <p>Thus, the sum could be $-2, 0$, or 2. Answer: E</p>
4.	<p>Solving $\frac{1}{m} + \frac{1}{n} = \frac{2}{2019}$ for n gives: $n = \frac{2019m}{2m - 2019}$. The smallest integer value for m that yields a positive integer for n is 1010, so that the denominator is 1 (any smaller integer value for m makes n negative). Answer: 1010</p>
5.	<p>$3!5!7! = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 1 \cdot 7! = (2 \cdot 5) \cdot (3 \cdot 3) \cdot (2 \cdot 4) \cdot 7! = 10 \cdot 9 \cdot 8 \cdot 7! = 10!$ Answer: 10</p>
6.	<p>Since $Area = \frac{1}{2} \cdot base \cdot height$, we can maximum the area by maximizing the base and height. If we allow one of the given sides to be the longest side, then the base or height would not be a maximum. Thus, neither side of length 4 or 5 can be the longest side. Letting the base be either 4 or 5, the height is maximized by making the base and height perpendicular. Thus, the Pythagorean theorem gives the third side (hypotenuse): $\sqrt{4^2 + 5^2} = \sqrt{41}$. Answer: $\sqrt{41}$</p> <p><u>Alternate Solution:</u> $Area = \frac{1}{2}ab\sin(\theta)$, where θ is the angle between the two given sides, with $\sin(\theta)$ maximized when $\theta = 90^\circ$. Now the third side can be obtained via the Pythagorean theorem as above.</p>
7.	<p>Let $x =$ the number of black socks, thus $10 - x =$ the number of white socks. The probability of selecting 2 black socks in succession is then $\frac{x}{10} \cdot \frac{x-1}{9}$. Hence, $\frac{x}{10} \cdot \frac{x-1}{9} = \frac{1}{3} \Rightarrow x^2 - x - 30 = 0 \Rightarrow (x-6)(x+5) = 0 \Rightarrow x = 6, -5$. Therefore, there are 6 black socks, and 4 white socks. The probability of <i>not</i> getting a matching pair (i.e. one black and 1 white, in either order) is:</p> $P(BW \text{ or } WB) = P(BW) + P(WB) = \frac{6}{10} \cdot \frac{4}{9} + \frac{4}{10} \cdot \frac{6}{9} = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}.$ <div style="text-align: right; border: 1px solid black; padding: 2px;">Answer: E</div>

8.	<p>I. $100!$ has $\frac{100}{2^1} = 50$ factors divisible by 2^1; $\frac{100}{2^2} = 25$ factors divisible by 2^2; $\frac{100}{2^3} = 12.5$, thus 12 factors divisible by 2^3, $\frac{100}{2^4} \approx 6.3$, thus 6 factors divisible by 2^4, $\frac{100}{2^5} \approx 3.1$, thus 3 factors divisible by 2^5, $\frac{100}{2^6} \approx 1.6$, thus 1 factor divisible by 2^6. Hence, $100!$ has $50 + 25 + 12 + 6 + 3 + 1 = 97$ 2's that factor it. Thus, 2^{97} is the greatest power of 2 that factors $100!$, making $\frac{100!}{2^{98}}$ not a whole number.</p> <p>II. $\frac{\log_{10}(\text{googol}^{\text{googol}})}{\log_{100}(\text{googolplex})} = \frac{\text{googol} \cdot \log_{10}(10^{100})}{\log_{100}(10^{\text{googol}})} = \frac{\text{googol} \cdot 100}{\text{googol} \cdot \log_{100}(10)} = \frac{10^{100} \cdot 100}{10^{100} \cdot (\frac{1}{2})} = 200$</p> <p>III. $\frac{\sqrt{1000}^{\text{googol}}}{\text{googolplex}} = \frac{\sqrt{100}^{\text{googol}} 10^{\text{googol}}}{10^{\text{googol}}} = \frac{\sqrt{100}^{\text{googol}} \cdot \sqrt{10}^{\text{googol}}}{10^{\text{googol}}} = \frac{10^{\text{googol}} \cdot 10^{\frac{1}{2} \cdot \text{googol}}}{10^{\text{googol}}} = 10^{\frac{1}{2} \cdot \text{googol}}$ $= 10^{5 \times 10^{99}}$ Clearly (now), II and III are whole numbers. Answer: A</p>
9.	<p>$w = \frac{1}{2} \cdot \text{googol} = \frac{1}{2} \cdot 10^{100} = 5 \times 10^{99}$, $x = \sqrt[100]{\text{googolplex}} = (10^{10^{100}})^{\frac{1}{100}} = 10^{10^{98}}$,</p> <p>$y = \ln(\text{googolplex}) = \ln(10^{\text{googol}}) = \text{googol} \cdot \ln(10) = \ln(10) \times 10^{100} \Rightarrow 2 \times 10^{100} < y < 3 \times 10^{100}$,</p> <p>For $z = \text{googol} \cdot \cos(\text{googol}^\circ)$, the question is "approximately what is $\cos(\text{googol}^\circ)$ and how does it compare to $\frac{1}{2}$ (to compare z to w)?" We thus need to know the reference angle of $(10^{100})^\circ$. $1000 \div 360 = 2$ with a remainder of 280. Hence, multiplying 1000 by 10 would give 20 with a "remainder" of 2800...but $2800 \div 360 = 7$ with a remainder of 280 as well. Therefore, repeated multiples of 10 will always leave a remainder of 280 when divided by 360. Hence, $(10^{100})^\circ$ is coterminal to 280°, which is in the 4th quadrant with a reference angle of 80°. Thus, $\cos(\text{googol}^\circ) = \cos(80^\circ)$ which is $\cos(80^\circ) < \cos(60^\circ) = \frac{1}{2}$, making $z < w$. Answer: z, w, y, x</p>
10.	<p>Letting r = the radius of the semicircle, and drawn to the hypotenuse of the triangle at the point of tangency, it forms a similar right triangle. By similar triangles, we obtain $\frac{r}{5-r} = \frac{2}{\sqrt{29}} \Rightarrow r = \frac{10}{2+\sqrt{29}}$ or $r = \frac{2}{5}(\sqrt{29}-2)$. Answer: $\frac{10}{2+\sqrt{29}} = \frac{2}{5}(\sqrt{29}-2)$</p> 
11.	<p>Let $f(x) = \alpha x + \beta$, where α and β are constants.</p> <p>I. Not true if f is a horizontal line (i.e. $\alpha = 0$).</p> <p>II. True, since $f(ax+b) = \alpha(ax+b) + \beta = \alpha ax + \alpha b + \beta$, also a linear function.</p> <p>III. True if $\alpha = 0$, making $f(\cos(x)) = 0 \cdot \cos(x) + \beta = \beta$ Answer: D</p>

12.	<p>The two triangles formed are similar, as they have corresponding angles. Letting h = the height of the smaller triangle, $1-h$ = the height of the larger triangle. By similar triangles we have: $\frac{h}{1/2} = \frac{1-h}{1} \Rightarrow 2h = 1-h \Rightarrow h = \frac{1}{3}$</p> <p>Thus, the height of the larger triangle is $\frac{2}{3}$, making the area $\frac{1}{2} \cdot 1 \cdot \frac{2}{3} = \frac{1}{3}$.</p>	
13.	<p>For all triangles, the sum of the two smaller sides must be greater than the longest side.</p> <p><u>Case I:</u> $\ln(x)$ is <i>not</i> the longest side, thus $\ln(4)$ must be.</p> $\Rightarrow \ln(3) + \ln(x) > \ln(4) \Rightarrow \ln(3x) > \ln(4) \Rightarrow 3x > 4 \Rightarrow x > \frac{4}{3}$ <p><u>Case II:</u> $\ln(x)$ is the longest side.</p> $\Rightarrow \ln(3) + \ln(4) > \ln(x) \Rightarrow \ln(12) > \ln(x) \Rightarrow 12 > x \Rightarrow x < 12$ <p>Hence, $\frac{4}{3} < x < 12$, making the only integer solutions: 2 through 11 (10 integers).</p>	<p style="text-align: right;">Answer: C</p> <p style="text-align: right;">Answer: 10</p>
14.	$\log_x(4) = \log_4(4x) \Rightarrow \frac{\log_4(4)}{\log_4(x)} = \log_4(4) + \log_4(x) \Rightarrow \frac{1}{\log_4(x)} = 1 + \log_4(x)$ $\Rightarrow 1 = \log_4(x) + \log_4^2(x) \Rightarrow \log_4^2(x) + \log_4(x) - 1 = 0.$ <p>Solving this via the quadratic formula gives: $\log_4(x) = \frac{1}{2}(-1 \pm \sqrt{5}) \Rightarrow x_1 = 4^{\frac{1}{2}(-1-\sqrt{5})} = 2^{-1-\sqrt{5}}$ and $x_2 = 4^{\frac{1}{2}(-1+\sqrt{5})} = 2^{-1+\sqrt{5}}$. Hence, the product of the solutions is: $2^{-1-\sqrt{5}} \cdot 2^{-1+\sqrt{5}} = 2^{-2} = \frac{1}{4}$.</p> <p><u>Alternate Solution:</u> From $\log_4^2(x) + \log_4(x) - 1 = 0$ (above), the sum of the roots is the negative coefficient of the $\log_4(x)$ term. Thus, $\log_4(x_1) + \log_4(x_2) = -1 \Rightarrow \log_4(x_1 x_2) = -1 \Rightarrow x_1 x_2 = 4^{-1}$.</p>	<p style="text-align: right;">Answer: C</p>
15.	<p>If the sine of the interior angles of a triangle are in the ratio $\sin(A) : \sin(B) : \sin(C) = 4 : 5 : 6$, then we also have $\frac{\sin(A)}{\sin(B)} = \frac{4}{5}$ and $\frac{\sin(B)}{\sin(C)} = \frac{5}{6}$. The Law of Sines then gives: $\frac{\sin(A)}{4} = \frac{\sin(B)}{5} = \frac{\sin(C)}{6}$, which means we can construct a triangle with sides $a = 4$, $b = 5$, and $c = 6$. Now we use the Law of Cosines to obtain:</p> $4^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cos(A) \Rightarrow \cos(A) = \frac{-45}{-60} = \frac{3}{4}$ $5^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cos(B) \Rightarrow \cos(B) = \frac{-27}{-48} = \frac{9}{16}$ $6^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos(C) \Rightarrow \cos(C) = \frac{-5}{-40} = \frac{1}{8}$ <p>Thus, $\cos(A) : \cos(B) : \cos(C) = \frac{3}{4} : \frac{9}{16} : \frac{1}{8} = 12 : 9 : 2$.</p>	<p style="text-align: right;">Answer: C</p>

16.	<p>Letting $\theta = \arccos(x)$ gives $\cos(\theta) = x$ and the accompanying right triangle. Also, $\theta = \arctan(x)$, which gives $\tan(\theta) = x$. From the triangle, we get $\tan(\theta) = \frac{\sqrt{1-x^2}}{x}$. Thus, $\frac{\sqrt{1-x^2}}{x} = x \Rightarrow x^2 = \sqrt{1-x^2}$, squaring both sides gives: $\frac{\sqrt{1-x^2}}{x} = x \Rightarrow x^4 = 1-x^2 \Rightarrow x^4 + x^2 - 1 = 0$. The quadratic formula gives $x^2 = \frac{-1 \pm \sqrt{5}}{2}$. For this problem $x^2 > 0$, so we take $x^2 = \frac{-1 + \sqrt{5}}{2} \Rightarrow x = \sqrt{\frac{\sqrt{5}-1}{2}}$ (taking the positive root since $\arccos(x) = \arctan(x)$ can only be true for $x > 0$).</p>	
17.	<p>The $4xy + y^2$ term suggests we can write part of the left hand side as a perfect square in terms of x and y. $13x^2 = 4x^2 + 9x^2$, so the equation can be written: $(4x^2 + 4xy + y^2) + (9x^2 - 6x + 1) = 0$. Which can be expressed as: $(2x + y)^2 + (3x - 1)^2 = 0$. Hence, $(2x + y)^2 = 0$ and $(3x - 1)^2 = 0$. This gives: $y = -2x$ and $x = \frac{1}{3} \Rightarrow x = \frac{1}{3}$ and $y = -\frac{2}{3}$.</p>	<div style="border: 1px solid black; padding: 5px; display: inline-block;">Answer: $\left(\frac{1}{3}, -\frac{2}{3}\right)$</div>
18.	<p>Letting $y = x - 4$ transforms the equation to: $(y + 3)(y + 1)(y - 1)(y - 3) = 20$. Grouping conjugate pairs: $(y + 3)(y - 3)(y + 1)(y - 1) = 20 \Rightarrow (y^2 - 9)(y^2 - 1) = 20 \Rightarrow y^4 - 10y^2 - 11 = 0$. This last expression factors: $(y^2 - 11)(y^2 + 1) = 0 \Rightarrow y = \pm\sqrt{11}$, ignoring the imaginary solutions. Hence, $x - 4 = \pm\sqrt{11} \Rightarrow x = 4 \pm \sqrt{11}$ with $x = 4 + \sqrt{11}$ being the larger real solution.</p>	<div style="border: 1px solid black; padding: 5px; display: inline-block;">Answer: $4 + \sqrt{11}$</div>
19.	<p>$\sin(k\pi x) = \frac{1}{2}$ when $k\pi x = \frac{\pi}{6} + 2\pi n$ or $k\pi x = \frac{5\pi}{6} + 2\pi n$, $n = 0, 1, 2, \dots$ (we want $n \geq 0$ so $k > 0$). Solving for x gives: $x_1 = \frac{1+12n}{6k}$ or $x_2 = \frac{5+12n}{6k}$. Since we seek solutions at $x = 4$, we get: $k_1 = \frac{1+12n}{24}$ and $k_2 = \frac{5+12n}{24}$. When $n = 0$, $k_1 = \frac{1}{24}$ yields one solution; when $n = 0$, $k_2 = \frac{5}{24}$ yields two solutions; when $n = 1$, $k_1 = \frac{13}{24}$ yields three solutions; when $n = 1$, $k_2 = \frac{17}{24}$ yields four solutions; when $n = 1$, $k_2 = \frac{25}{24}$ yields five solutions. Hence, $\frac{17}{24} \leq k < \frac{25}{24}$.</p> <p><u>Note:</u> The interval <i>includes</i> $k = \frac{17}{24}$ so that the 4th solution occurs at $x = 4$, while we must <i>exclude</i> $k = \frac{25}{24}$ so the 5th solution "just misses" being included at $x = 4$.</p>	<div style="border: 1px solid black; padding: 5px; display: inline-block;">Answer: $k \in \left[\frac{17}{24}, \frac{25}{24}\right)$</div>
20.	<p>Letting B = the number of boys in the family, and G = the number of girls in the family... Each girl has $G - 1$ sisters and B brothers, giving ① $B = G - 1$. Each boy has G sisters and $B - 1$ brothers, giving ② $G = 2(B - 1)$. Solving ① and ② gives $B = 3$ and $G = 4$.</p>	<div style="border: 1px solid black; padding: 5px; display: inline-block;">Answer: 7</div>