

# New York State Mathematics Association of Two-Year Colleges

## Math League Contest ~ Fall 2022

**Directions:** You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must be given in *exact* form, i.e. in terms of fractions, radicals,  $\pi$ , etc.

1. Let  $f(x) = \frac{x}{2^x - 1} + \frac{x}{2}$  and  $g(x) = \frac{x}{2^x - 1} - \frac{x}{2}$ , for all real  $x \neq 0$ . Which of the following statements are/is true?  
 I.  $f$  is an even function.      II.  $g$  is an even function.      III.  $f \cdot g$  is an even function.  
 A) I only      B) II only      C) III only      D) I and II only      E) I and III only

2.  $\frac{\frac{\frac{1}{1+1}}{1+1+1}}{\frac{1+1+1+1}{1+1+1+1}} = ?$       A)  $\frac{1}{576}$       B)  $\frac{1}{64}$       C)  $\frac{1}{9}$       D)  $\frac{1}{4}$       E) 1

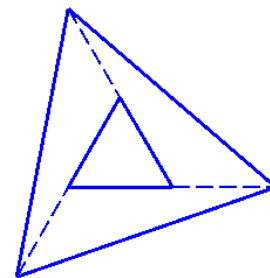
3. In a certain month, three Tuesdays fall on even-numbered dates. Which day of the week is the 28<sup>th</sup> day of the month?  
 A) Friday      B) Monday      C) Sunday      D) Tuesday  
 E) There is not enough information to uniquely determine the day of the week.

4. Placed in numerical order (smallest to largest):  $x = \frac{3}{3002002}$ ,  $y = \frac{5}{5003009}$ ,  $z = \frac{7}{7005001}$  is which of the 6 choices:  $x, y, z$  or  $x, z, y$  or  $y, x, z$  or  $y, z, x$  or  $z, x, y$  or  $z, y, x$ ?

5. In the following sum, each letter represents a unique digit. Thus,  $BB$  is a 2-digit number and  $CCC$  is a 3-digit number. What digit is represented by  $A$ ?

$$\begin{array}{r} A \\ A \\ + BB \\ \hline CCC \end{array}$$

6. Given an equilateral triangle with sides of length 1, a new equilateral triangle is formed by extending the three sides by 1 unit then connecting the new endpoints, as shown. What is the ratio of the area of the new triangle to the area of the original (smaller) one, in simplest form?



Problems 7 and 8 refer to the following cross-number puzzle. All answers are 3-digit numbers, so that none have a zero in the hundreds position.

Across

1. A permutation of the digits of a number that is both a perfect square and a perfect cube.
2. A perfect 4<sup>th</sup> power.
3. A perfect cube.

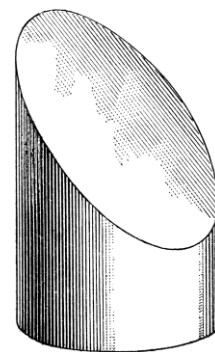
Down

1. A number.
2. An odd number whose reverse is a prime number.
3. A perfect 4<sup>th</sup> power.

1	2	3
2		
3		

7. What is "3 Across"?
8. What is "1 Down"?
9. An artist is planning on mixing together any number of different colors from her palette. A mixture results as long as the artist combines at least two colors. If the number of possible mixtures is less than 500, what is the greatest number of colors the artist could have in her palette?
10. A certain whole number between 2000 and 2100 has exactly 15 factors. What is the number?  
For example: 10 has exactly 4 factors, namely 1, 2, 5, and 10.

11. A circular cylinder with a diameter of 1 has a plane cut made at an angle of 60° to the base, giving an elliptical cross-section, as shown. If the ellipse is then placed on the  $xy$ -plane with its center at the origin and major axis along the  $x$ -axis, then what is its equation?

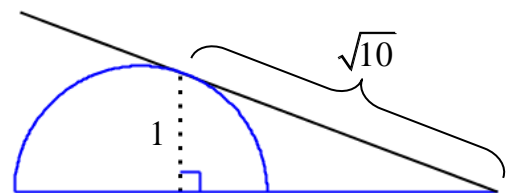


- A)  $x^2 + 4y^2 = 1$                       B)  $x^2 + 3y^2 = 4$                       C)  $2x^2 + \sqrt{3}y^2 = 4$   
D)  $2x^2 + 3y^2 = 4$                       E)  $3x^2 + 4y^2 = 1$

12. What is the maximum value of the function  $f(x) = \frac{2022}{\log(4x^2 - 4x + 101)}$ , where  $x \in (-\infty, \infty)$  and the logarithm is base 10?

- A)  $\frac{2022}{\log(101)}$                       B) 1011                      C) 2022                      D) 4044  
E) There is no maximum value, the function is unbounded.

13. In triangle  $ABC$ ,  $\cos(3A - B) + \sin(A + 3B) = 2$ . What is the *degree* measure of angle  $C$ ?
14. If  $(x + 2)^2 + (y - 4)^2 = 45$  for real numbers  $x$  and  $y$ , then what is the maximum value of  $x^2 + y^2$ ?
15. When the equation  $\log_x(y) + 2022 = 2023\log_y(x)$  is solved for  $y$  in terms of  $x$ , two solutions are obtained. The product of the two solutions is  $x^n$ , where  $n$  is an integer. What is  $n$ ?
16. For the quadratic equation  $x^2 + bx + r = 0$ , the values for  $b$  and  $r$  are determined by the roll of dice. Two ordinary 6-sided dice (with the numbers 1 through 6), one blue and one red, are rolled. The number showing up on the blue die is assigned to  $b$ , and the number showing up on the red die is assigned to  $r$ . What is the probability the resulting quadratic equation will have at least one real solution?
- A)  $\frac{1}{3}$       B)  $\frac{5}{12}$       C)  $\frac{1}{2}$       D)  $\frac{19}{36}$       E)  $\frac{7}{12}$
17. If 2022 is divided by 1, 2, 3, and so on up to 1000, then what is the largest remainder that can be obtained? For example, 2022 divided by 100 yields a remainder of 22, but that is not the largest possible remainder.
18. The graph of  $y^3 - y^2 + y + x^2y - x^2 = 1$  is
- A) a point.      B) a line.      C) a parabola.  
D) symmetric about the origin (i.e. an odd function).      E) not a function.
19. The diagram shows a semicircle with a line resting on it (i.e. it is tangent to the semicircle). The dotted line represents the distance from the base of the semicircle to the point of tangency, and has length 1. The distance from the point of tangency to the point where the line intersects the extension of the base is  $\sqrt{10}$ , as shown. What is the length of the radius of the semicircle?

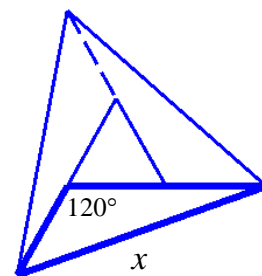


20. An inhabitant of Logic Land either speaks the truth all day or lies all day, but can never tell the truth and a lie on the same day. Which of the following statements can such a person *never* say?
- A) Yesterday, I told the truth.      B) Yesterday, I lied.      C) Today, I tell the truth.  
D) Today, I lie.      E) Tomorrow, I will lie.

## Math League Contest ~ Fall 2022 ~ Solutions

1.	$f(x) = \frac{x}{2^x - 1} + \frac{x}{2} = \frac{x(2^x + 1)}{2(2^x - 1)} \Rightarrow f(-x) = \frac{-x(2^{-x} + 1)}{2(2^{-x} - 1)} = \frac{-x(1 + 2^x)}{2(1 - 2^x)} = \frac{x(2^x + 1)}{2(2^x - 1)} = f(x),$ <p>so <math>f</math> is an even function.</p> $g(x) = \frac{x}{2^x - 1} - \frac{x}{2} = \frac{x(3 - 2^x)}{2(2^x - 1)} \Rightarrow g(-x) = \frac{-x(3 - 2^{-x})}{2(2^{-x} - 1)} = \frac{x(3 \cdot 2^x - 1)}{2(2^x - 1)} \neq g(x).$ <p>Thus, <math>g</math> is not an even function. Since <math>g</math> is not even, <math>f \cdot g</math> is not even.</p>	Answer: A
2.	$\frac{\frac{1}{1+1}}{\frac{1+1+1}{1+1+1+1}} = \frac{\left(\frac{1}{2}\right)}{\frac{1+1+1}{1+1+1+1}} = \frac{\left(\frac{1}{2}\right)}{\frac{3}{4}} = \frac{\left(\frac{1}{2}\right)}{\frac{4}{4}} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{8}{8}\right)} = \frac{1}{24} \cdot \frac{3}{8} = \frac{1}{64}$	Answer: B
3.	<p>In order for three Tuesdays of a month to fall on even-numbered days, the first Tuesday of the month must land on an even-numbered day (otherwise only two Tuesdays could fall on even-numbered days). Certainly the first Tuesday of the month can fall on the 2<sup>nd</sup>, so that the subsequent Tuesdays would land on the 9<sup>th</sup>, 16<sup>th</sup>, 23<sup>rd</sup>, and 30<sup>th</sup> – giving three Tuesdays falling on even-numbered days. Can the first Tuesday fall on the 4<sup>th</sup> of the month? If so, then the subsequent Tuesdays would have to land on the 11<sup>th</sup>, 18<sup>th</sup>, 25<sup>th</sup>, and then the 32<sup>nd</sup>. Since no month has more than 31 days, the only way three Tuesdays can fall on even-numbered days is as first described. Hence, the 28<sup>th</sup> day of the month must be a Sunday.</p>	Answer: C
4.	<p>Rather than compare the fractions by obtaining a common denominator, which would be quite cumbersome to do by hand, we can obtain a common numerator (a much easier task to perform by hand) and then compare denominators. The <i>least common numerator</i> is <math>3 \cdot 5 \cdot 7 = 105</math>. Thus,</p> $x = \frac{3}{3002002} \cdot \frac{5 \cdot 7}{5 \cdot 7} = \frac{3 \cdot 35}{3002002 \cdot 35} = \frac{105}{105070070}, \quad y = \frac{5}{5003009} \cdot \frac{3 \cdot 7}{3 \cdot 7} = \frac{5 \cdot 21}{5003009 \cdot 21} = \frac{105}{105063189},$ <p>and <math>z = \frac{7}{7005001} \cdot \frac{3 \cdot 5}{3 \cdot 5} = \frac{7 \cdot 15}{7005001 \cdot 15} = \frac{105}{105075015}</math>.</p> <p>The smallest is the one with the largest denominator, followed by the one with the second largest denominator, then the one with the smallest denominator. Thus, from smallest to largest we have: <math>z, x, y</math>.</p>	Answer: z, x, y
5.	<p>Clearly, <math>C = 1</math> since the sum of two 1-digit numbers and a 2-digit number cannot exceed <math>9 + 9 + 99 = 117</math>. <math>B</math> must be either 8 or 9, because if <math>B</math> were 7 or less, the sum of <math>BB</math> and two 1-digit numbers would yield only a 2-digit number. If <math>B = 8</math>, then <math>A + A = 23</math> so that <math>A + A + 88 = 111</math>. However, <math>A + A</math> cannot be odd. Hence, <math>B = 9</math> with <math>A = 6</math>.</p>	Answer: 6

6. The area of an equilateral triangle with sides of length  $s$  is  $\frac{\sqrt{3}}{4}s^2$ . Thus, the area of the original (smaller) triangle is  $\frac{\sqrt{3}}{4}$ . Let  $x$  be the length of the sides of the new (larger) triangle. Analyzing one of the three new obtuse triangles formed (in bold), we can determine  $x$ . Using the Law of Cosines, we get:  $x^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cos(120^\circ) \Rightarrow x^2 = 5 - 4(-\cos(60^\circ)) = 5 - 4(-\frac{1}{2})$ . Thus,  $x^2 = 5 + 2 = 7$ , making the area of the new (larger) triangle  $\frac{\sqrt{3}}{4} \cdot 7$ . Hence, the



ratio of the areas is:  $\frac{\frac{\sqrt{3}}{4} \cdot 7}{\frac{\sqrt{3}}{4}} = 7$ .

Answer: 7

#### Alternate Solution

Take one of the three obtuse triangles that surround the original triangle, using the side length of 2 as the base, and extend the base so we can form the height (both shown as dotted lines). The height of this triangle is the same as the height of the original equilateral triangle, but this triangle has a base of 2 (twice that of the original). Thus, the area is 2 times the area of the original triangle. The new larger triangle consists of three of these obtuse triangles plus the original one. Hence, the new larger triangle has an area that is  $3 \cdot 2 + 1 = 7$  times the original.



#### 7. Across

1. If a number is both a perfect square and a perfect cube, then it must be a perfect 6<sup>th</sup> power. The only 3-digit perfect 6<sup>th</sup> power is 729 (i.e.  $3^6 = 27^2 = 9^3 = 729$ ). This answer is a permutation of the digits 7, 2, and 9.
2. The only 3-digit perfect 4<sup>th</sup> powers are:  $4^4 = 256$  and  $5^4 = 625$  ( $3^4 = 81$  and  $6^4 = 1296$ ). Thus, this answer is either 256 or 625.
3. The 3-digit perfect cubes are:  $5^3 = 125$ ,  $6^3 = 216$ ,  $7^3 = 343$ ,  $8^3 = 512$ , and  $9^3 = 729$ .

#### Down

1. Not helpful.
2. Helpful later.
3. From 2 across, we see the only choices are 256 and 625. However, it cannot be 625, since that would make 2 across have a "2" in the units digit. Thus, 3 down must be 256, making 2 across 625. This gives a "6" in the units digit of 3 across, making it 216. From 1 across, we know only the digits "7" and "9" remain. However, the reverse of 2 down is prime, so it cannot be 921, since its reverse is not prime ( $129 = 3 \cdot 43$ ). Hence, 2 down must be 721 (with 127 being prime), forcing a "9" in the only remaining position.

1	2	3
9	7	2
2	6	2
	5	5
3	2	1
	6	6

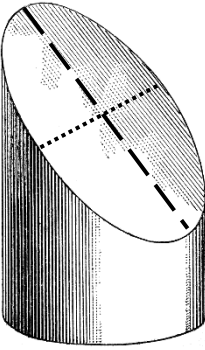
Answer: 216

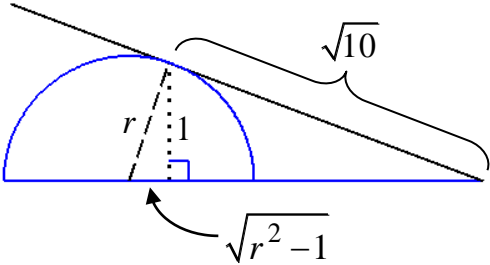
8. From above, we get 962.

Answer: 962

9. Let  $n$  be the number of colors in the artist's palette. There are  $2^n$  possible mixtures, since there are 2 choices for each color (it may be used or not used). However, this includes not using any color (1 way) and using only one color ( $n$  ways). The closest power of 2 to 500 is  $2^9 = 512$ . Subtracting the 1 way of using no colors and the 9 ways of using only one color, gives 502 mixtures, which slightly exceeds 500. Hence, 8 colors must be the maximum.

Answer: 8

10.	<p>Only perfect squares have an odd number of factors, since non-perfect squares have unique pairs of factors whose product yields the number. Thus, we seek perfect squares between 2000 and 2100. Observing that <math>44^2 &lt; 2000</math>, <math>45^2 = 2025</math>, and <math>46^2 &gt; 2100</math>, 2025 is the only perfect square between 2000 and 2100. Thus, 2025 is the only number between 2000 and 2100 that has an odd number of factors (15 of them).</p>	<p style="text-align: right;"><b>Answer: 2025</b></p>
11.	<p>The dotted line corresponds to the minor axis of the ellipse, which has a length equal to the diameter of the cylinder, making it 1. The dashed line corresponds to the major axis of the ellipse. The major axis is the hypotenuse of a right triangle with a base of 1 (the diameter of the cylinder) and a <math>60^\circ</math> angle at the base. Thus, the hypotenuse, i.e. major axis, has a length of <math>\frac{1}{\cos(60^\circ)} = \frac{1}{\frac{1}{2}} = 2</math>. An ellipse centered at the origin has the form <math>\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1</math>; here <math>a = \frac{2}{2} = 1</math>, and <math>b = \frac{1}{2}</math>. Thus, the equation becomes: <math>\frac{x^2}{1^2} + \frac{y^2}{(\frac{1}{2})^2} = 1 \Rightarrow x^2 + 4y^2 = 1</math>.</p>	 <p style="text-align: right;"><b>Answer: A</b></p>
12.	<p><math>\frac{2022}{\log(4x^2 - 4x + 101)}</math> is maximized when the denominator is the smallest positive value it can achieve. The quadratic <math>4x^2 - 4x + 101</math> is minimized for <math>x = -\frac{-4}{2 \cdot 4} = \frac{1}{2}</math> (i.e. on the axis of symmetry), giving the minimum <math>4(\frac{1}{2})^2 - 4(\frac{1}{2}) + 101 = 1 - 2 + 101 = 100</math>. Thus, the denominator has a minimum value of <math>\log(100) = 2</math>. Therefore, <math>f_{\max} = \frac{2022}{2} = 1011</math>.</p>	<p style="text-align: right;"><b>Answer: B</b></p>
13.	<p>Since the maximum values of the cosine and sine functions are 1, the only way <math>\cos(3A - B) + \sin(A + 3B) = 2</math> is for <math>\cos(3A - B) = 1</math> and <math>\sin(A + 3B) = 1</math>. Thus, ① <math>3A - B = 0^\circ</math> and ② <math>A + 3B = 90^\circ</math>. Solving ① and ②, yields <math>A = 9^\circ</math> and <math>B = 27^\circ</math>. Hence, angle <math>C</math> must be <math>180^\circ - (9^\circ + 27^\circ) = 144^\circ</math>.</p>	<p style="text-align: right;"><b>Answer: 144°</b></p>
14.	<p>For any point <math>(x, y)</math> in the plane, the distance from the origin is <math>\sqrt{x^2 + y^2}</math>. Thus, maximizing <math>x^2 + y^2</math> is equivalent to maximizing the distance <math>(x, y)</math> is from the origin. Hence, the problem can be restated: <i>What is the maximum distance between the origin and a point on the circle given by <math>(x+2)^2 + (y-4)^2 = 45</math>?</i> The points on a circle that are closest and furthest from the origin lie on the line through the origin and the center of the circle. The line through <math>(0,0)</math> and <math>(-2,4)</math> is <math>y = -2x</math>. This line and the circle intersect where: <math>(x+2)^2 + (-2x-4)^2 = 45 \Rightarrow 5(x+2)^2 = 45 \Rightarrow (x+2)^2 = 9 \Rightarrow x+2 = \pm 3 \Rightarrow x = -5</math> or <math>x = 1</math>. This gives the points on the circle at extreme distances from the origin: <math>(-5,10)</math> and <math>(1,-2)</math>, with <math>(-5,10)</math> being the furthest. Therefore, the maximum value of <math>x^2 + y^2</math> is <math>(-5)^2 + 10^2 = 125</math>.</p> <p><u>Alternate (Intuitive) Solution</u></p> <p>The center of the circle, <math>(-2,4)</math>, is <math>\sqrt{(-2)^2 + 4^2} = \sqrt{20}</math> units from the origin. The radius of the circle is <math>\sqrt{45}</math>. Thus, the maximum distance (discussed above) is <math>\sqrt{20} + \sqrt{45} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}</math>. Hence, the maximum value of <math>x^2 + y^2</math> is <math>(5\sqrt{5})^2 = 125</math>.</p>	<p style="text-align: right;"><b>Answer: 125</b></p>

15.	<p>Since <math>\log_y(x) = \frac{\log_b(x)}{\log_b(y)} = \left(\frac{\log_b(y)}{\log_b(x)}\right)^{-1} = (\log_x(y))^{-1} = \frac{1}{\log_x(y)}</math>, the equation can be written as <math>\log_x(y) + 2022 = 2023 \frac{1}{\log_x(y)}</math>. Multiplying by <math>\log_x(y)</math> and rearranging gives: <math>(\log_x(y))^2 + 2022 \log_x(y) - 2023 = 0</math>. The last equation factors, giving: <math>(\log_x(y) - 1)(\log_x(y) + 2023) = 0</math>. Thus, <math>\log_x(y) - 1 = 0 \Rightarrow y = x</math> or <math>\log_x(y) + 2023 = 0 \Rightarrow y = x^{-2023}</math>, making the product <math>x \cdot x^{-2023} = x^{-2022}</math>. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: -2022</span></p>
16.	<p><math>x^2 + bx + r = 0</math> will have real solutions only when <math>b^2 - 4 \cdot 1 \cdot r \geq 0</math>, from the quadratic formula. Thus, we need to determine the probability that <math>b^2 \geq 4r</math>. Since there are 6 outcomes for each roll of each die, there are a total of <math>6 \cdot 6 = 36</math> possible outcomes for the combined roll. Now we merely need to count how many of those 36 rolls will have <math>b^2 \geq 4r</math>. When <math>r = 1</math>, <math>b</math> could be 2, 3, 4, 5, or 6; when <math>r = 2</math>, <math>b</math> could be 3, 4, 5, or 6; when <math>r = 3</math>, <math>b</math> could be 4, 5, or 6; when <math>r = 4</math>, <math>b</math> could be 4, 5, or 6; when <math>r = 5</math>, <math>b</math> could be 5, or 6; and when <math>r = 6</math>, <math>b</math> could be 5, or 6. This gives 19 ways out of the 36 possible rolls (with each outcome equally likely). Thus, the probability is <math>\frac{19}{36}</math>. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: D</span></p>
17.	<p>If the divisors were unlimited, then we should choose 2023 or larger, so the remainder would be 2022. However, 1000 is our maximum divisor. Thus, we must select a divisor that divides 2022 almost a whole number of times. Half of 2022 is 1011, so we would like to divide by 1012 to obtain a remainder of 1010, but 1012 is too big (over 1000). The next best choice would be a divisor that divides 2022 almost 3 times. Since <math>2022 \div 3 = 674</math>, we should divide by 675, which yields a remainder of 672 as the maximum. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: 672</span></p>
18.	<p><math>y^3 - y^2 + y + x^2y - x^2 = 1 \Rightarrow y^3 - y^2 + y - 1 + x^2y - x^2 = 0 \Rightarrow (y^3 - y^2) + (y - 1) + (x^2y - x^2) = 0</math>  Factoring by grouping: <math>y^2(y - 1) + (y - 1) + x^2(y - 1) = 0 \Rightarrow (y - 1)(y^2 + 1 + x^2) = 0</math>, and since <math>y^2 + 1 + x^2 \geq 1</math>, the last equation can only be satisfied when <math>y - 1 = 0</math>, i.e. when <math>y = 1</math>. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: B</span></p>
19.	<p>After drawing the radius (the dashed line) to the point of tangency, we see that the new (small) right triangle with the radius as the hypotenuse is similar to the larger right triangle with hypotenuse <math>\sqrt{10}</math>. The similar triangles relationship gives: <math>\frac{\sqrt{r^2 - 1}}{r} = \frac{1}{\sqrt{10}} \Rightarrow \frac{r^2 - 1}{r^2} = \frac{1}{10} \Rightarrow r^2 = \frac{10}{9} \Rightarrow r = \frac{\sqrt{10}}{3}</math>. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: <math>\frac{\sqrt{10}}{3}</math></span></p> 
20.	<p>Statements A, B, C, and E can all be said by the person whether lying or not. If the person is telling the truth, then statement D would be a lie; if the person is lying, then statement D would be true. Hence, statement D can never be said by an inhabitant of Logic Land. <span style="float: right; border: 1px solid black; padding: 2px;">Answer: D</span></p>