New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2023

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. If $x^2 + \frac{1}{x^2} = 2023$ and we let $w = \left| x + \frac{1}{x} \right|$, then which of the following is true about *w*?

   A) $42 \le w < 43$ B) $43 \le w < 44$ C) $44 \le w < 45$ D) $45 \le w < 46$ E) $46 \le w < 47$

2. Burger King recently had an advertisement where they claimed "There are 221,184 ways to have a Whopper." Assume this number is correct and ignore quantities like "extra" or "light" (e.g. you can either have pickles or not have pickles, you can have either ketchup or no ketchup, "extra" or "light" amounts are not an option). Also, assume you cannot combine sauces (e.g. you can have either ketchup or mayonnaise, but not both). If *n* represents the number of available sauces, then which of the following is not a possible value for *n*?

   A) 2 B) 3 C) 4 D) 5 E) None, they are all *possible*.

3. If $f(1-2x) = \frac{x}{2-x}$, then what is $f(x)$?

4. 1000 days after January 1, 2024 will be in September 2026. What will be the date?

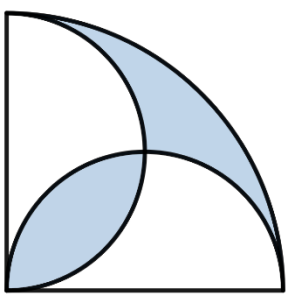
5. If $\sec(x) + \tan(x) = 2023$, then what is the value of $\sec(x) - \tan(x)$?

6. What is the 2023rd digit to the right of the decimal point in the decimal expansion of $\frac{4}{41}$?

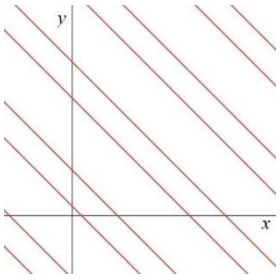
   A) 0 B) 5 C) 6 D) 7 E) 9

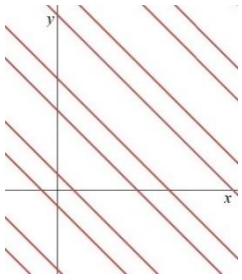
7. A ¼-circle of radius 2 has two semicircles of radius 1 inscribed, as shown. What is the area of the shaded region?

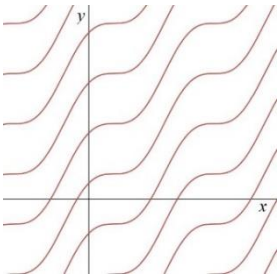
   A) 1 B) $\frac{\pi}{3}$ C) $\frac{\pi - 1}{2}$ D) $\pi - 2$ E) $\frac{\pi}{2}$

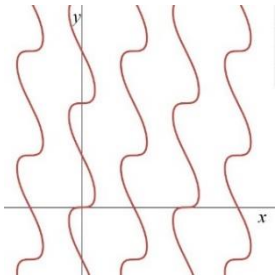


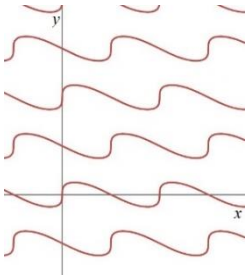
8. Which of the following graphs is represented by the equation $\sin(x+y)=\frac{1}{2}$?

A) 

B) 

C) 

D) 

E) 

9. Alice scored 15 baskets out of 20 attempts in the first half of a basketball game, and 10 out of 10 in the second half. Bianca took 12 shots in the first half and 18 in the second half of the game. In each half, Alice made a higher percentage of her attempts than Bianca. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Bianca score in the second half than in the first?

A) 7 B) 8 C) 9 D) 10 E) This is an impossible situation.

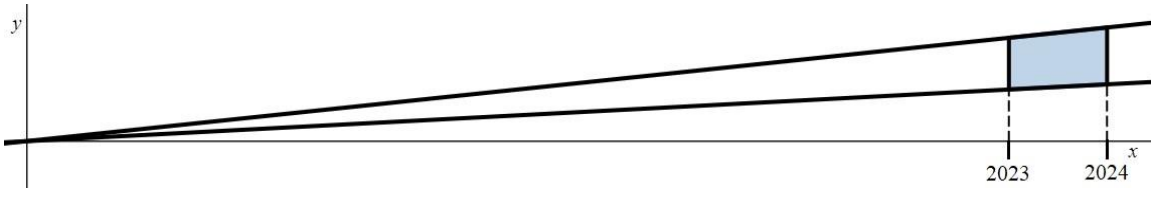
10. Ten marbles numbered 1 to 10 are in a bag. Brian reaches into the bag and randomly removes one of the marbles. Then Chris reaches into the bag and randomly removes a marble. What is the probability that the sum of the two marbles drawn will be even? Assume each marble is equally likely to be selected.

A) $\frac{4}{9}$ B) $\frac{9}{19}$ C) $\frac{1}{2}$ D) $\frac{10}{19}$ E) $\frac{5}{9}$

11. What is the value of $\sum_{n=0}^{90}\cos^2(n°)=\cos^2(0°)+\cos^2(1°)+\cos^2(2°)+\cos^2(3°)+\ldots+\cos^2(90°)$?

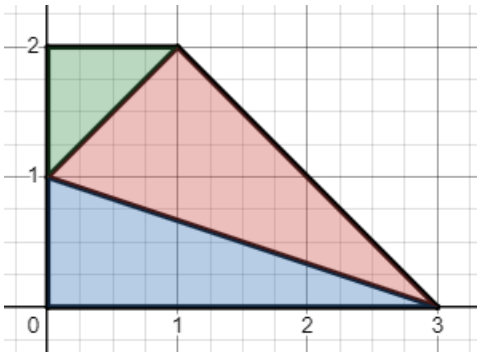
A) 14π B) 44.5 C) 45 D) 45.5 E) 15π

12. The diagram (not drawn to scale) shows two linear functions, $y=mx$ and $y=2mx$ with the shaded region being the area bound by them between $x=2023$ and $x=2024$. If the area of that region is 1, then what is the value of *m*?



A) $\frac{1}{4047}$ B) $\frac{1}{2024}$ C) $\frac{2}{4047}$ D) $\frac{1}{2023}$ E) $\frac{4047}{(2023)(2024)}$

13. What is $\arctan(1)+\arctan(2)+\arctan(3)$?
Hint: Use the accompanying diagram.
A) 165° B) 180° C) 195°
D) 210° E) 225°



14. Which of these large numbers: 1.4^{8000}, 2^{4000}, 3^{3000}, 5^{2000}, 26^{999}, is the largest?
A) 1.4^{8000} B) 2^{4000} C) 3^{3000} D) 5^{2000} E) 26^{999}

15. What value of *n* satisfies the equation: $\log_2(3)\cdot\log_3(4)\cdot\log_4(5)\cdot\ldots\cdot\log_n(n+1)=2023$?
Note: The " … " means the pattern continues through $\log_n(n+1)$.

16. How many solutions, *x*, are there for the equation: $\left(x^2-x-3\right)^{x^2+2x-10}=1$?
A) 1 B) 2 C) 3 D) 4 E) 5

17. Lattice points are points that have integer coordinates, e.g. $(1,-3)$, $(-2,0)$, and $(7,8)$, but not $(\frac{1}{2},1)$.
How many lattice points that are on the line $y=-\frac{1}{6}x+\frac{2023}{3}$ are in the first quadrant?
Note: Points on the *x*-axis or *y*-axis are not in a quadrant, coordinates need to be non-zero.

18. Joe and Erin play the following game: Joe will toss one coin and Erin will toss two coins. If Joe gets the same number of heads (i.e. "heads up") as Erin, he wins. What is the probability that Joe wins? Assume that the coins are *fair*, i.e. the probability of "heads up" on any one toss is ½.

19. An *n*-sided regular polygon has 350 diagonals. What is the value of *n* (i.e. how many sides must this polygon have)? Examples: A square (4-sided) has 2 diagonals, and a pentagon (5-sided) has 5 diagonals).

20. Four cards are lying on a table as shown. Each card has a letter on one side and a whole number on the other side. Josh said, "If a vowel is on one side of any card, then an even number is on the other side." Elizabeth showed Josh was wrong by turning over only one card. Which card did Elizabeth turn over?

P	Q	2	3

A) P B) Q C) 2 D) 3
E) It is impossible to determine from the given information.

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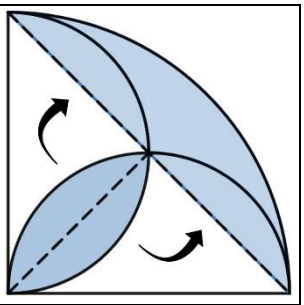
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2023 ~ Solutions

1.	$w=\sqrt{\left(x+\frac{1}{x}\right)^2}=\sqrt{x^2+1+1+\frac{1}{x^2}}=\sqrt{x^2+\frac{1}{x^2}+2}=\sqrt{2023+2}=\sqrt{2025}=45$ Answer: D
2.	For n available sauces (with at most only one allowed), there are $n+1$ choices (since not having any sauce is yet another option). Factoring 221,184 gives $2^{13}3^3$. If $n=2$, then $n+1=3$ must be a factor, which it is. If $n=3$, then $n+1=4$ must be a factor, which it is. If $n=5$, then $n+1=6$ must be a factor, which it is. If $n=4$, then $n+1=5$ must be a factor, which it is not. Thus, there cannot be exactly 4 available sauces. Answer: C
3.	Replacing x with the inverse of $1-2x$, i.e. $\frac{x-1}{-2}$, will give the desired result since $f\left(1-2\left(\frac{x-1}{-2}\right)\right)=f(x)$. Hence, $f(x)=\frac{\left(\frac{x-1}{-2}\right)}{2-\left(\frac{x-1}{-2}\right)}=\frac{x-1}{-4-(x-1)}=\frac{x-1}{-3-x}=\frac{1-x}{3+x}$. Answer: $\frac{1-x}{3+x}$
4.	Since $1000=365+365+270$, we need only determine the date 270 days after January 1, 2026. 30 days later will be January 31, 28 more days (for a total of 58 days) will be February 28, 31 more days (89 total) make it March 31, add 30 more (119 total) to get to April 30, 31 more (150 total) for May 31, 30 more (180 total) for June 30, 31 more (211 total) to get to July 31, another 31 days (242 total) for August 31. Now we need to add 28 additional days to get 270 days after January 1, 2026, which makes the date September 28. However, we did not take into account that 2024 is a leap year, having 29 days in February. Thus, the date will be September 27 rather than the 28th. Answer: 27
5.	Multiplying by $\frac{\sec(x)-\tan(x)}{\sec(x)-\tan(x)}=1$, gives $\frac{\sec(x)+\tan(x)}{1}\cdot\frac{\sec(x)-\tan(x)}{\sec(x)-\tan(x)}=2023 \Rightarrow$ $\frac{\sec^2(x)-\tan^2(x)}{\sec(x)-\tan(x)}=2023$, but one form of the Pythagorean Identity is $1+\tan^2(x)=\sec^2(x)$ or $1=\sec^2(x)-\tan^2(x)$. Thus, $\frac{\sec^2(x)-\tan^2(x)}{\sec(x)-\tan(x)}=2023$ becomes $\frac{1}{\sec(x)-\tan(x)}=2023$. Taking the reciprocal of both sides yields: $\sec(x)-\tan(x)=\frac{1}{2023}$. Answer: $\frac{1}{2023}$
6.	$\frac{4}{41}=0.\overline{09756}$, has a cycle of length 5 and 2023 divided by 5 leaves a remainder of 3. Thus, the 2023rd digit to the right of the decimal point will be the 3rd digit in the cycle – which is "7." Answer: D

7. The two shaded segments at the right angle can be "moved" to fill in the empty segments as shown. Thus, the area we seek is the area of the ¼-circle of radius 2 minus the area of the isosceles right triangle with legs of length 2. Therefore, the area is $\frac{1}{4}\pi \cdot 2^2 - \frac{1}{2}\cdot 2\cdot 2 = \pi - 2.$ Answer: D



8. $\sin(x+y)=\frac{1}{2} \Rightarrow \sin(x+y+2\pi k)=\frac{1}{2} \Rightarrow y=\sin^{-1}\left(\frac{1}{2}\right)-x-2\pi k,\ k=0,\pm1,\pm2,\pm3,\ldots$

$\Rightarrow y=\frac{\pi}{6}-x-2\pi k$ or $y=-x+\frac{\pi}{6}-2\pi k,$ which is a family of parallel lines with slopes of $-1.$

Thus, we must choose between choices A and B. When $k=0$, the y-intercept of the line $y=-x+\frac{\pi}{6}$ is $\frac{\pi}{6}$; when $k=1$, the y-intercept of the line $y=-x+\frac{\pi}{6}-2\pi$ is $\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$. Therefore, the graph is not symmetric about the origin (as in choice B). Hence, A is the best choice. Answer: A

9. Let $x=$ the number of baskets Bianca scored in the 1st half, and $y=$ the number of baskets Bianca scored in the 2nd half. In each half Alice had a higher percentage (or ratio) of shots made gives the following inequalities: ① $\frac{15}{20}>\frac{x}{12}$ and ② $\frac{10}{10}>\frac{y}{18}$. They finished with the same overall percentage/ratio gives the equation: $\frac{15+10}{20+10}=\frac{x+y}{12+18} \Rightarrow$ ③ $x+y=25.$

The inequalities simplify to: ① $x<9$ and ② $y<18$. Solving ③ for y gives: $y=25-x$. Substituting that into ② gives $25-x<18 \Rightarrow x>7$. $x<9$ and $x>7$ tells us that $x=8$ and $y=17$. Therefore, Bianca made $17-8=9$ more baskets in the 2nd half than in the 1st. Answer: C

10. The sum of the two numbers on the marbles will be even only if they have the same parity (i.e. both even or both odd). Suppose Brian selects an even-numbered marble, which has a probability of $\frac{1}{2}$. Then Chris must also select an even-numbered marble, which has a probability of $\frac{4}{9}$. This makes the probability of them both selecting even-numbered marbles $\frac{1}{2}\cdot\frac{4}{9}=\frac{2}{9}$. The same probabilities occur for them both selecting odd-numbered marbles, also giving $\frac{2}{9}$. Hence, the probability of either situation is $\frac{2}{9}+\frac{2}{9}=\frac{4}{9}$. Answer: A

Alternate Solution: There are ${}_{10}C_2=45$ ways of selecting the 2 marbles from the 10 available. Then there are $5\cdot4=20$ ways for the parity of the two numbers to match, giving the probability $\frac{20}{45}=\frac{4}{9}$.

11. Since $\cos(n°)=\sin(90°-n°)$, we can write:

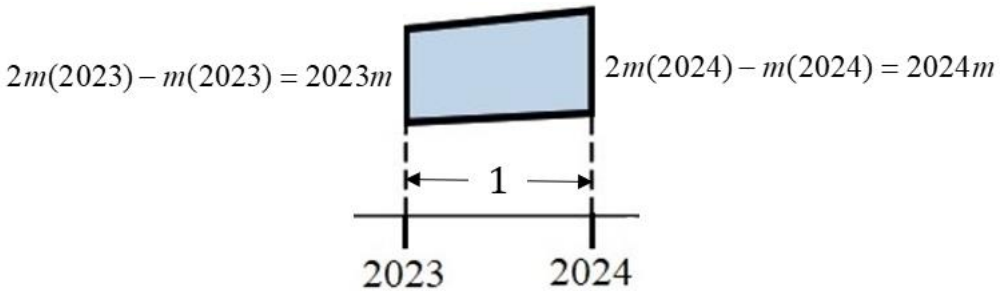
$$\sum_{n=0}^{90}\cos^2(n°)=\sum_{n=0}^{44}\cos^2(n°)+\cos^2(45°)+\sum_{n=46}^{90}\cos^2(n°)=\sum_{n=0}^{44}\cos^2(n°)+\left(\frac{\sqrt{2}}{2}\right)^2+\sum_{n=46}^{90}\sin^2(90°-n°)$$

$$=\sum_{n=0}^{44}\cos^2(n°)+\left(\frac{1}{2}\right)+\sum_{n=0}^{44}\sin^2(n°)=\sum_{n=0}^{44}\cos^2(n°)+\sum_{n=0}^{44}\sin^2(n°)+\left(\frac{1}{2}\right)$$

$$=\sum_{n=0}^{44}\left(\cos^2(n°)+\sin^2(n°)\right)+\left(\frac{1}{2}\right)=\sum_{n=0}^{44}(1)+\left(\frac{1}{2}\right)=45+\frac{1}{2}=\frac{91}{2}=45.5$$

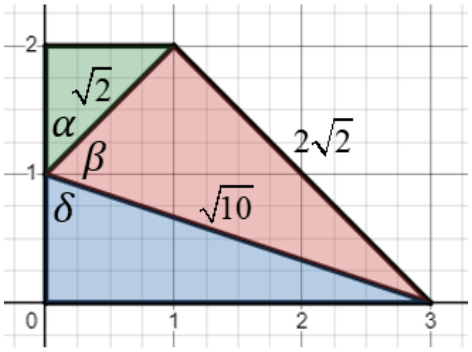
Answer: D

12. The shaded region is a trapezoid with bases (i.e. parallel sides) of length $2023m$ and $2024m$, with a "height" (i.e. distance between the bases) of 1, as shown. Thus, the area is $\frac{1}{2}(2023m+2024m)\cdot 1$, which is equal to 1. This gives: $\frac{1}{2}(4047m)=1 \Rightarrow m=\frac{2}{4047}$.



Answer: C

13. The Pythagorean Theorem gives the hypotenuse of the two smaller triangles as $\sqrt{2}$ and $\sqrt{10}$, and the distance formula yields the distance between points $(1,2)$ and $(3,0)$ is $2\sqrt{2}$. Now we can verify, by the Pythagorean Theorem, that the largest triangle is a right triangle, since $\left(\sqrt{2}\right)^2+\left(2\sqrt{2}\right)^2=\left(\sqrt{10}\right)^2$. Thus, $\tan(\alpha)=\frac{1}{1}=1$, $\tan(\beta)=\frac{2\sqrt{2}}{\sqrt{2}}=2$, and $\tan(\delta)=\frac{3}{1}=3$. Which gives: $\alpha=\arctan(1)$, $\beta=\arctan(2)$, and $\delta=\arctan(3)$. However, $\alpha+\beta+\delta=180°$. Hence, $\arctan(1)+\arctan(2)+\arctan(3)=180°$.



Answer: B

14. $1.4^{8000}=\left(1.4^2\right)^{4000}=1.96^{4000}<2^{4000}$

$2^{4000}=\left(2^4\right)^{1000}=16^{1000}<25^{1000}=\left(5^2\right)^{1000}=5^{2000}<27^{1000}=\left(3^3\right)^{1000}=3^{3000}$

and $26^{999}<27^{1000}=3^{3000}$

This gives the ordering: $1.4^{8000}<2^{4000}<25^{1000}$, $26^{999}<3^{3000}$

Answer: C

15. Since $\log_b(x)=\frac{\ln(x)}{\ln(b)}$, we can write the equation as:

$\frac{\ln(3)}{\ln(2)}\cdot\frac{\ln(4)}{\ln(3)}\cdot\frac{\ln(5)}{\ln(4)}\cdot\ldots\cdot\frac{\ln(n)}{\ln(n-1)}\cdot\frac{\ln(n+1)}{\ln(n)}=2023$, which simplifies to

$\frac{\ln(n+1)}{\ln(2)}=2023 \Rightarrow \log_2(n+1)=2023 \Rightarrow n+1=2^{2023} \Rightarrow n=2^{2023}-1$

Answer: $2^{2023}-1$

16. In order for $\left(x^2-x-3\right)^{x^2+2x-10}=1$ to have a solution, we need

(i) $x^2-x-3=1$ and $x^2+2x-10$ can be any value $\Rightarrow x^2-x-4=0 \Rightarrow x=\frac{1\pm\sqrt{17}}{2}$, or

(ii) $x^2+2x-10=0$ with $x^2-x-3\neq 0 \Rightarrow x=-1\pm\sqrt{11}$ (in each case $x^2-x-3\neq 0$), or

(iii) finally, $x^2-x-3=-1$ with $x^2+2x-10$ even $\Rightarrow x^2-x-2=0 \Rightarrow x=-1, 2$.

But, only $x=2$ yields an even number for $x^2+2x-10$.

Thus, there are 5 solutions to the equation.

Answer: E

17.	Solving $y = -\frac{1}{6}x + \frac{2023}{3}$ for x gives: $x = 4046 - 6y$. The range of y-values is from 1 to 674, since if $y \geq 675,\ x < 0$. Thus, there are 674 lattice points in the first quadrant that are on the line $y = -\frac{1}{6}x + \frac{2023}{3}$. Answer: 674
18.	The only way Joe can win is if Erin gets no heads or only 1 head (as Joe cannot get 2 heads). $\text{P(Erin gets 0 Heads)} = \frac{1}{4}$ and $\text{P(Joe gets 0 Heads)} = \frac{1}{2}$, making $\text{P(Both get 0 Heads)} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. $\text{P(Erin gets 1 Head)} = \frac{1}{2}$ and $\text{P(Joe gets 1 Head)} = \frac{1}{2}$, making $\text{P(Both get 1 Head)} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Thus, the probability Joe wins is $\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$. Answer: 3/8
19.	Each of the n vertices of a regular polygon can be connected to any other vertex to form a diagonal except itself and the two adjacent vertices (because it would trace an edge), which gives $n(n-3)$ diagonals. However, this counts each diagonal twice (e.g. vertex A to vertex C and vertex C to vertex A). Thus, for an n-sided regular (or convex) polygon, there are $\frac{1}{2}n(n-3)$ possible diagonals. Therefore, we must solve $\frac{1}{2}n(n-3) = 350$. $\Rightarrow \frac{1}{2}n(n-3) = 350 \Rightarrow n^2 - 3n - 700 = 0 \Rightarrow (n+25)(n-28) = 0 \Rightarrow n = -25,\ 28$ But, n must be an integer greater than 2. Hence, $n = 28$. Answer: 28
20.	Turning over card P or Q would tell us nothing about Josh's claim. Turning over card 2 couldn't prove Josh's statement false. If the other side has a vowel on it, it would corroborate the claim. If there is consonant, then Josh's statement does not apply. The only way Elizabeth could have disproven Josh's statement is if there is a card that has a *vowel on one side and an odd number on the other*. Therefore, Elizabeth must have turned over card 3 and there had to be a vowel on the other side. Answer: D

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