New York State Mathematics Association of Two-Year Colleges

NYSMATYC 1967

Math League Contest ~ Fall 2024

<u>Directions</u>: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. If $f(x)$ is an odd function and $4f(x) - f(-x) = \tan(x)$, then what is the value of $f\left(\frac{\pi}{3}\right)$?

   A) $\frac{\sqrt{3}}{5}$ B) $\frac{\sqrt{3}}{3}$ C) $\sqrt{3}$ D) $\frac{5}{\sqrt{3}}$ E) $5\sqrt{3}$

2. What is half of 2^{2024}?

   A) $2^{\sqrt{2024}}$ B) 2^{1012} C) $2^{2024} - 2^{2023}$ D) $2^{2023.5}$ E) $2^{2024} - 2^{1012}$

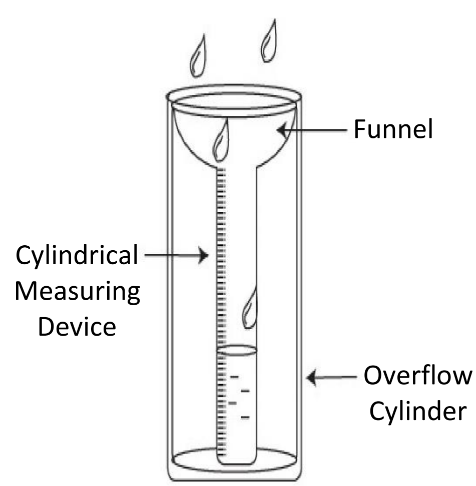
3. Augustus De Morgan was a British mathematician and logician who was born and died in the 1800's. He once wrote, "On my birthday I was x years old in the year x^2." What year was he born?

4. If this triangular array is continued with the same pattern, with two more consecutive integers being added to each new row, then what would be the number directly below 2024?

			1			
		2	3	4		
	5	6	7	8	9	
10	11	12	13	14	15	16
			⋮			

5. A simple rain gauge (i.e. a device for measuring rainfall) can be made by attaching a funnel to the top of a cylindrical measuring device (e.g. a graduated cylinder), as shown but not drawn to scale. If the diameter of the top of the funnel is 6 inches and the diameter of the cylinder is 3 inches, then what should the calibration scale be on the cylinder? In other words, how far apart should the graduations on the cylinder be so each interval corresponds to 1 inch of rain?

   A) $\frac{1}{4}$ inch B) $\frac{1}{2}$ inch C) 1 inch
   D) 2 inches E) 4 inches



6. The number of intersection points of the graph of $y = \frac{1}{x}$ and $y = mx + b$, where $m > 0$ and b is any real number, is

A) 0 B) 1 C) 2 D) 3 E) It depends on the value of *b*.

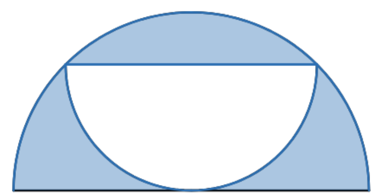
7. What is the distance between the line $y = 2x + 2$ and the point $(2, 0)$?

A) $\frac{6}{\sqrt{5}}$ B) $3\sqrt{\frac{5}{6}}$ C) $2\sqrt{2}$ D) $\frac{5}{\sqrt{3}}$ E) 3

8. The product of the squares of the first 1000 prime numbers ends in how many zeros? (i.e. How many trailing zeros does $2^2 \cdot 3^2 \cdot \ldots \cdot P_{1000}^2$ have, where P_{1000} is the thousandth prime number?)

9. If the mean and median are equal for the five numbers: 10, 7, 13, 4, and x, then what is the sum of all possible values for *x*?

10. A semicircle with a smaller semicircle is inscribed so that the diameters are parallel, as shown. Let A_1 equal the area of the smaller semicircle, the unshaded region; and A_2 equal the area of the shaded region, the area of the region between the smaller semicircle and the larger one. What is $\frac{A_1}{A_2}$?



A) $\frac{\sqrt{2}}{2}$ B) $\sqrt{\frac{2}{3}}$ C) $\frac{\sqrt{3}}{2}$ D) 1 E) $\frac{\pi}{3}$

11. $2^{28} - 2^8$ has exactly two distinct factors in the 90's. What is the difference of those two factors?

A) 2 B) 3 C) 4 D) 5 E) 8

12. $\sin^{-1}(\sin(10)) = ?$ Note: The 10 is in radians.

A) $10 - 3\pi$ B) $\frac{1}{10}$ C) $3\pi - 10$ D) $10 - 2\pi$ E) 10

13. What integer value for *n* makes the following relationship true: $n < \log\left(\frac{1}{2024}\right) < n + 1$?

14. The following numbers: $\sqrt[3]{2}$, $\sqrt[5]{3}$, $5^{2/15}$ placed in numerical order, i.e. from smallest to largest, are

A) $\sqrt[3]{2}, \sqrt[5]{3}, 5^{2/15}$ B) $\sqrt[3]{2}, 5^{2/15}, \sqrt[5]{3}$ C) $\sqrt[5]{3}, \sqrt[3]{2}, 5^{2/15}$

D) $5^{2/15}, \sqrt[3]{2}, \sqrt[5]{3}$ E) $5^{2/15}, \sqrt[5]{3}, \sqrt[3]{2}$

15. What is the domain of the function $f(x) = \log_{2024}\left(36 - 9x + 4x^3 - x^4\right)$?

16. Consider a 7-game series between two sports teams (e.g. the MLB World Series or the NBA Finals), i.e. the team that wins 4 games wins the series. Assume the two teams are equally matched, i.e. they each have a probability of ½ of winning any one game. The probability the series goes to 7 games is equal or closest to

A) $\frac{1}{10}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{1}{2}$

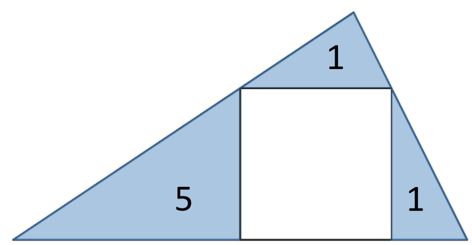
17. What is the sum of all solutions to $|4x+1| - |3x-1| = 2x+1$?

18. If both of the roots of the function $f(x) = x^2 + 25x + 2024$ are increased by 1, then the resulting numbers will be the roots of which function?

A) $g(x) = x^2 + 23x + 2000$ B) $g(x) = x^2 + 27x + 2000$ C) $g(x) = x^2 + 23x + 2025$

D) $g(x) = x^2 + 27x + 2025$ E) $g(x) = x^2 + 23x + 2049$

19. The diagram, not drawn to scale, shows a square (unshaded) inscribed in a triangle forming three smaller triangles (shaded), with the areas of the three triangles shown. What is the area of the inscribed square?



20. The fictitious country of Logicland consists of truthtellers who always tell the truth, and liars who always lie. A married couple (a husband and a wife) in Logicland are asked, “Which of you, if either, is a truthteller and which of you, if either, is a liar?” The husband promptly responds, “We are both liars.” Based only on this information, correctly classify the couple.

A) The husband is a truthteller and the wife is a liar.
B) The husband is a liar and the wife is a truthteller.
C) The husband and wife are both liars.
D) The husband and wife are both truthtellers.
E) It is impossible to determine from the given information.

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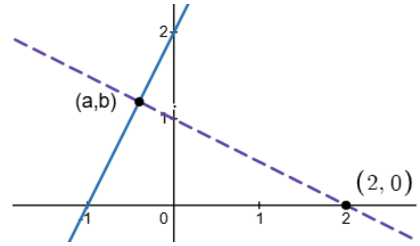
Math League Contest ~ Fall 2024 ~ Solutions

1.	$f(x)$ being an odd function means $f(-x)=-f(x)$. Thus, $4f(x)-f(-x)=\tan(x)$ is equivalent to $4f(x)+f(x)=\tan(x) \Rightarrow 5f(x)=\tan(x) \Rightarrow f(x)=\frac{1}{5}\tan(x)$. Thus, $f\left(\frac{\pi}{3}\right)=\frac{1}{5}\tan\left(\frac{\pi}{3}\right)=\frac{1}{5}\sqrt{3}$. Answer: A
2.	Half of 2^{2024} is $\frac{2^{2024}}{2}=2^{2024-1}=2^{2023}$. Clearly choices A, B, D, and E are not equivalent. Choice C can be simplified by factoring out 2^{2023}: $2^{2024}-2^{2023}=2^{2023}\left(2^1-1\right)=2^{2023}\cdot 1=2^{2023}$. Answer: C
3.	Since $42^2=1764$, $43^2=1849$, and $44^2=1936$, $x=43$ is the only viable option (being that De Morgan lived his entire life in the 1800's). Thus, he must have been born in $43^2-43=1806$. Answer: 1806
4.	Notice that the last number in each row is the square of the row number, i.e. the 4 in row 2 is 2^2, the 9 in row 3 is 3^2, and so on. In which row will 2024 be? Since $44^2=1936$ will be the last number in the 44th row and $45^2=2025$ will be the last number in the 45th row, we know 2024 will be the 2nd to last number in the 45th row. Hence, the number directly below 2024 in the 46th row will be the 3rd to last number in that row, which is $46^2-2=2116-2=2114$. Answer: 2114
5.	Since the radius of the funnel opening is 2 times the radius of the cylindrical measuring device, the cross-sectional area of the funnel opening is $2^2=4$ times that of the cylinder. Thus, the funnel will collect 4 times as much rain as would just the cylinder. Which means 1 inch of rain collected by the funnel will produce a 4 inch rise in the cylinder. Therefore, the scaling on the cylinder needs to be 4 times greater than what is collected by the funnel. Answer: E
6.	Setting them equal to each other gives: $mx+b=\frac{1}{x} \Rightarrow mx^2+bx-1=0$. By the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4m(-1)}}{2m}=\frac{-b\pm\sqrt{b^2+4m}}{2m}$. Since $m>0$, $b^2+4m>0$ and x will take on two real distinct values. Hence, there will be two points of intersection. We can also deduce the two points of intersection by considering the graphs and realizing any line with a positive slope will intersect the hyperbola at exactly two points. Answer: C

7. The distance between two objects is the shortest distance, which is the perpendicular measure. Thus, we seek the distance along the perpendicular line (the dashed line in the diagram) between (2,0) and the point of intersection between the two lines, given by (a,b) in the diagram. The perpendicular line to $y = 2x + 2$ has a slope of –½. Which gives $y - 0 = -\frac{1}{2}(x-2)$ or $y = -\frac{1}{2}x + 1$. $y = 2x+2$ and $y = -\frac{1}{2}x + 1$ intersect at $\left(-\frac{2}{5}, \frac{6}{5}\right)$. The distance between $(2,0)$ and $\left(-\frac{2}{5}, \frac{6}{5}\right)$ is $\sqrt{\left(2-\left(-\frac{2}{5}\right)\right)^2 + \left(0-\frac{6}{5}\right)^2} = \sqrt{\left(\frac{12}{5}\right)^2 + \left(-\frac{6}{5}\right)^2} = \sqrt{\frac{180}{25}} = \sqrt{\frac{36}{5}} = \frac{6}{\sqrt{5}}$.



Answer: A

8. Each trailing zero ("zero at the end") of a whole number corresponds to a factor of 10. Since $10 = 2 \cdot 5$ we need a 2 and a 5 as a factor for each trailing zero. 2 is the only even prime number, and 5 is the only prime number that is a multiple of 5. Therefore, only the prime numbers 2 and 5 will yield zeros in the product of primes. Being that each is squared, we get $2^2 \cdot 5^2 = 100$ multiplying all the other primes squared. Hence, there will be only 2 trailing zeros.

Answer: 2

9. In numerical order: 4, 7, 10, and 13. There are three cases to consider:

I. If $x \geq 10$: Then the median is 10 and the mean will be $\frac{4+7+10+13+x}{5} = \frac{34+x}{5}$, which must equal 10. Thus, $\frac{34+x}{5} = 10$ which gives $x = 16$.

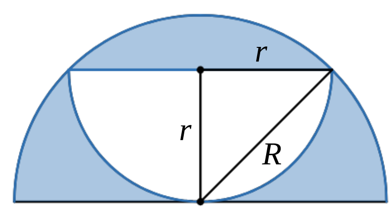
II. If $7 \leq x < 10$: Then the median is x and the mean must equal $x \Rightarrow \frac{34+x}{5} = x$, which yields $x = \frac{17}{2} = 8.5$.

III. If $x < 7$: Then the median is 7 and the mean must equal 7 $\Rightarrow \frac{34+x}{5} = 7$, and so $x = 1$.

Thus, the sum of the possible x-values is $16 + 8.5 + 1 = 25.5$ or $\frac{51}{2}$.

Answer: 25.5 or $\frac{51}{2}$

10. Letting R representing the radius of the larger semicircle and r the radius of the smaller semicircle, the diagram gets labeled as shown. Thus, the triangle formed is a 45º-45º -90º (right) triangle. Which makes $R = \sqrt{2}r$. Therefore, the area of the smaller semicircle is $\frac{1}{2}\pi r^2$ and the shaded area is $\frac{1}{2}\pi\left(\sqrt{2}r\right)^2 - \frac{1}{2}\pi r^2 = \pi r^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi r^2$.



Hence, the two areas are equal making the ratio 1.

Answer: D

11. $2^{28} - 2^8 = 2^8\left(2^{20}-1\right) = 2^8\left(2^{10}-1\right)\left(2^{10}+1\right) = 2^8\left(2^5-1\right)\left(2^5+1\right)\left(2^{10}+1\right) = 2^8(31)(33)(1025)$

$= 2^8(31)(3\cdot 11)(5\cdot 205) = 2^8(31)(3\cdot 11)(5\cdot 5\cdot 41)$

Thus, the prime factorization is $2^8 \cdot 3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$. The only way to obtain factors in the 90's from this is: $3 \cdot 31 = 93$ and $2^5 \cdot 3 = 32 \cdot 3 = 96$. Hence, the difference is $96 - 93 = 3$.

Answer: B

12.	Since $-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$, $\sin^{-1}(\sin(10)) \ne 10$. We need to find the reference angle of 10 radians and know if $\sin(10)$ is positive or negative. Since 10 radians is only slightly larger than 3π radians, 10 radians must be in the 3rd quadrant. Thus, the reference angle is $10-3\pi$ and $\sin(10)$ is negative. Therefore, $\sin^{-1}(\sin(10)) < 0$ and must be $-(10-3\pi) = 3\pi - 10$. Answer: C
13.	$\log\left(\frac{1}{2024}\right) = \log\left(2024^{-1}\right) = -\log(2024)$ Also, since $10^3 = 1000$ and $10^4 = 10000$, $\log(2024)$ must be between 3 and 4. Which tells us that $-4 < -\log(2024) < -3$. Hence, $n = -4$. Answer: -4
14.	$\sqrt[3]{2} = 2^{\frac{1}{3}}$ and $\sqrt[5]{3} = 3^{\frac{1}{5}}$ Now we raise all three numbers to the 15th power to eliminate the fractional exponents: $\left(2^{\frac{1}{3}}\right)^{15} = 2^5 = 32$, $\left(3^{\frac{1}{5}}\right)^{15} = 3^3 = 27$, and $\left(5^{\frac{2}{15}}\right)^{15} = 5^2 = 25$. Since each of the original numbers are positive, raising them to a positive power preserves their ordering. Therefore, since $25 < 27 < 32$, $5^{2/15} < \sqrt[5]{3} < \sqrt[3]{2}$. Answer: E
15.	The domain of $f(x) = \log_{2024}\left(36 - 9x + 4x^3 - x^4\right)$ is where $36 - 9x + 4x^3 - x^4 > 0$. $36 - 9x + 4x^3 - x^4$ factors by grouping: $36 - 9x + 4x^3 - x^4 = 9(4-x) + x^3(4-x) = (4-x)\left(9+x^3\right)$ Thus, we need $(4-x)\left(9+x^3\right) > 0$. Solving where $(4-x)\left(9+x^3\right) = 0$ gives $x = 4$ and $x = -\sqrt[3]{9} = -3^{2/3}$. Now we just need to check the sign of $(4-x)\left(9+x^3\right)$ for $x < -\sqrt[3]{9}$, $-\sqrt[3]{9} < x < 4$, and $x > 4$. For $x < -\sqrt[3]{9}$ (e.g. $x = -3$), $(4-x)\left(9+x^3\right) < 0$. For $-\sqrt[3]{9} < x < 4$ (e.g. $x = 0$), $(4-x)\left(9+x^3\right) > 0$. For $x > 4$ (e.g. $x = 5$), $(4-x)\left(9+x^3\right) < 0$. Therefore, the domain is $x \in \left(-\sqrt[3]{9}, 4\right)$ or $-\sqrt[3]{9} < x < 4$. Answer: $x \in \left(-\sqrt[3]{9}, 4\right)$
16.	In order for such a series to get to a game 7, the first 6 games must be split 3 games each. Since we are assuming the chance of either team winning any one game is $\frac{1}{2}$, the probability of any one of the many ways to get a 3-3 tie after 6 games is $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$. Without loss of generality, let's use the recent World Series matchup between the New York Yankees and Los Angeles Dodgers, using "Y" for a Yankees win and "D" for a Dodgers win. We need to count the number of ways there can be 3 "Y's" and 3 "D's" (i.e. YYYDDD, YYDYDD, etc.), i.e. the number of 6-letter permutations there are using 3 "Y's" and 3 "D's." The answer is $\frac{6!}{3!\,3!}$, since there are 6 letters with repetitions of 3 "Y's" and 3"D's." $\frac{6!}{3!\,3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot3\cdot2\cdot1} = 5\cdot4 = 20$. Thus, the probability is $20\cdot\frac{1}{64} = \frac{20}{64} = \frac{10}{32} \approx \frac{1}{3}$. Answer: D

17. To solve $|4x+1|-|3x-1|=2x+1$ for x, let's consider the cases where the absolute values transition, i.e. where they go from changing the sign to not changing the sign. That is where $4x+1=0$ and where $3x-1=0$, giving $x=-\frac{1}{4}$ and $x=\frac{1}{3}$. Thus, we need to consider the three cases:

I. Where $x \le -\frac{1}{4}$: The equation becomes $-(4x+1)-\left(-(3x-1)\right)=2x+1 \Rightarrow x=-1$.

II. Where $-\frac{1}{4} < x \le \frac{1}{3}$: The equation becomes $(4x+1)-\left(-(3x-1)\right)=2x+1 \Rightarrow x=\frac{1}{5}$.

III. Where $x > \frac{1}{3}$: The equation becomes $(4x+1)-(3x-1)=2x+1 \Rightarrow x=1$.

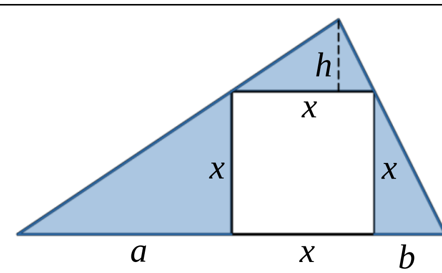
Hence, the sum of all solutions is $-1+\dfrac{1}{5}+1=\dfrac{1}{5}$.

Answer: $\dfrac{1}{5}$

18. If a quadratic equation, with a coefficient of 1 for the x^2 term, has roots r_1 and r_2, then its equation is $f(x)=(x-r_1)(x-r_2)=x^2-(r_1+r_2)x+r_1r_2$. Since $f(x)=x^2+25x+2024$, $r_1+r_2=-25$ and $r_1r_2=2024$. The new quadratic with roots r_1+1 and r_2+1 will have equation $g(x)=\left(x-(r_1+1)\right)\left(x-(r_2+1)\right)$. After multiplying and collecting like-terms, the new quadratic becomes $g(x)=x^2-(r_1+r_2+2)x+r_1r_2+r_1+r_2+1$. Substituting $r_1+r_2=-25$ and $r_1r_2=2024$ into $g(x)$ gives: $g(x)=x^2-(-25+2)x+2024+(-25)+1=x^2+23x+2000$.

Answer: A

19. Letting x be the side lengths of the square, a the length of the base of the large lower left triangle, and b the length of the base of the lower right triangle, as shown. The area of the triangles can now be expressed in terms of variables.



The large/left-most triangle: ① $\frac{1}{2}ax=5 \Rightarrow ax=10$

The small/right-most triangle: ② $\frac{1}{2}bx=1 \Rightarrow bx=2$

The small/top triangle: ③ $\frac{1}{2}hx=1 \Rightarrow hx=2$, where h is the height of the triangle (illustrated by the dashed line). From equations ① and ② we get $a=5b$. From equations ② and ③ we get $h=b$ and $b=\frac{2}{x}$. Thus, the base of the original triangle is $a+x+b=5b+x+b=6b+x=\frac{12}{x}+x$, and its height is $x+h=x+b=x+\frac{2}{x}$. This makes the area of the original triangle $\frac{1}{2}\left(\frac{12}{x}+x\right)\left(x+\frac{2}{x}\right)$, but this must equal the sum of the areas of the three smaller triangles and the square, which is $5+1+1+x^2=7+x^2$. Equating these gives $\frac{1}{2}\left(\frac{12}{x}+x\right)\left(x+\frac{2}{x}\right)=7+x^2$. Multiplying through by $2x^2$ to eliminate the fractions gives $\left(12+x^2\right)\left(x^2+2\right)=14x^2+2x^4$. Expanding this last result gives $x^4+14x^2+24=14x^2+2x^4$, which reduces to $x^4=24$. Taking the square root of both sides, and taking the positive root, yields $x^2=\sqrt{24}$ – which is the area of the square.

Answer: $\sqrt{24}$

20. If the husband is a truthteller, then him stating that he is a liar would be a false statement – contradicting him being a truthteller. Thus, the husband must be a liar. If he is a liar, then the only way his statement "We are both liars." would be false is for his wife to be a truthteller. Therefore, the husband must be a liar and his wife a truthteller.

Answer: B