NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2024

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. 84% of x is 65% of y, and 91% of y is 56% of z. What fraction of z is x?

   A) $\frac{10}{21}$ B) $\frac{672}{845}$ C) $\frac{14}{15}$ D) $\frac{15}{14}$ E) $\frac{21}{10}$

2. If $2024(x-2)^4 - 2023(x-1)^3 + 2022x^2 - 2021(x+1) + 2020 = ax^4 + bx^3 + cx^2 + dx + e$, then what is the value of $a+b+c+d+e$?

3. In the year 3000, January will have exactly four Tuesdays and exactly four Saturdays. On which day of the week will January 1, 3000 fall?

4. The numbers $\sqrt[4]{2}$, $\sqrt[6]{3}$, $\sqrt[9]{5}$ arranged from smallest to largest (left to right) is:

   A) $\sqrt[4]{2}, \sqrt[6]{3}, \sqrt[9]{5}$ B) $\sqrt[4]{2}, \sqrt[9]{5}, \sqrt[6]{3}$ C) $\sqrt[6]{3}, \sqrt[4]{2}, \sqrt[9]{5}$ D) $\sqrt[6]{3}, \sqrt[9]{5}, \sqrt[4]{2}$ E) $\sqrt[9]{5}, \sqrt[4]{2}, \sqrt[6]{3}$

5. For a certain integer, n, $\frac{(n+2)!}{(n-2)!} = 421200$. What is the value of n?

   Note: The factorial is defined: $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$, for example $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

6. The equation $100^x + 100^{1-x} = 25$ has two distinct real solutions, call them x_1 and x_2 with $x_1 < x_2$. What is $\frac{x_2}{x_1}$?

   A) $\log_4(5)$ B) $\log_{25}(100)$ C) $\log_4(10)$ D) $\log_5(20)$ E) $\log_2(5)$

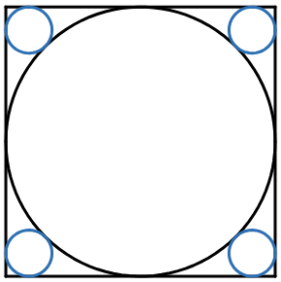
7. Which of the following is equivalent to $\sqrt{\sin^4(x)+4\cos^2(x)}$?

A) $\sin^2(x)+2\cos(x)$ B) $\sin(x)+2\cos(x)$ C) $2-\sin(2x)$ D) $2-\sin^2(x)$ E) $1+|\cos(x)|$

8. The equation $\log_{2023}(x)\cdot\log_{2024}(x)=1$ has two distinct real solutions. Let x_1 be the smaller of the two solutions. If n is an integer such that $n<x_1<n+1$, then what is the value of n?

9. In a class with boys and girls, there are 10 girls. If two students are selected at random to form a team, the probability that both are girls is $\frac{3}{7}$. How many boys are in the class?

10. A circle of radius 1 is inscribed in a square with sides of length 2. In each corner of the square a small circle is inscribed so they are tangent to two sides of the square and the circle, as shown. What is the radius of each of the four small circles?



A) $1-\frac{\sqrt{3}}{2}$ B) $\frac{2-\sqrt{2}}{4}$ C) $3-2\sqrt{2}$ D) $\frac{\sqrt{2}}{8}$ E) $\frac{\sqrt{2}-1}{2}$

11. If $\log_{\frac{1}{2}}(x)+\log_2(x)+\log_4(x)+\log_8(x)=1$, then what is the value of x?

12. Express the sum $\sum_{n=1}^{2024} ni^n = i+2i^2+3i^3+4i^4+\ldots+2024i^{2024}$ in $a+bi$ form. Where $i\equiv\sqrt{-1}$.

13. If $f\left(\frac{x+1}{x+2}\right)=\frac{x}{3x+4}$, then what is the value of $f(1)$?

A) -2 B) $\frac{1}{7}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) The value is undefined.

14. If $x^2+4x+1=0$, then what is the value of $x^3+\frac{1}{x^3}$?

15. If $\frac{\sin(\theta)}{x}=\frac{\cos(\theta)}{y}$, then what is $\sin^2(\theta)-\cos^2(\theta)$?

A) $\frac{x^2-y^2}{x^2+y^2}$ B) $\frac{y^2-x^2}{y^2+x^2}$ C) x^2-y^2 D) y^2-x^2 E) $1-x^2y^2$

16. The *Lambert W* function, also known as the product logarithm function, is the inverse function of $f(x) = xe^x$ and is often denoted by $W(x)$, i.e. $W(x) = f^{-1}(x)$. Solving $x + e^x = 2024$ for x in terms of W gives:

A) $\ln(2024) - W(2024)$ B) $2024 - W\left(e^{2024}\right)$ C) $\ln(2024) + W(2024)$

D) $2024 - W(2024)$ E) $2024 + W\left(e^{2024}\right)$

17. Which of the following statements imply that the positive real number x is rational?

I. $\sqrt{x}$ is rational.

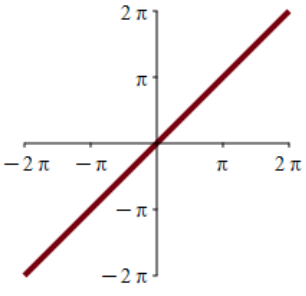
II. x^2 and x^3 are rational.

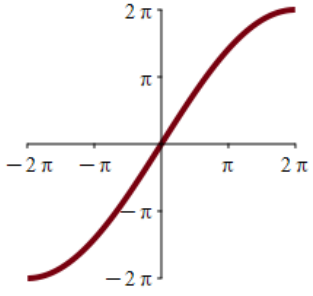
III. x^4 and x^6 are rational.

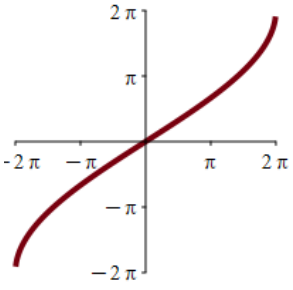
IV. x^{-1} is rational.

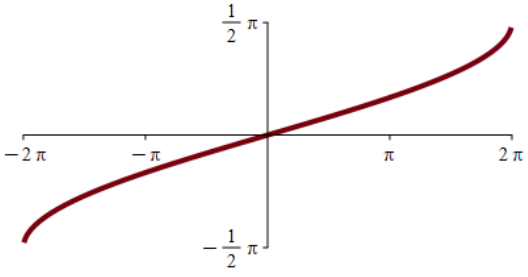
A) I only.
B) I and II only.
C) I and IV only.
D) II and III only.
E) I, II, and IV only.

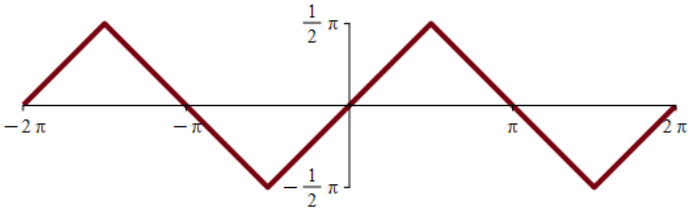
18. Which of the following graphs best represents the graph of $y = \arcsin(\sin(x))$ for $x \in [-2\pi, 2\pi]$?

A) 

B) 

C) 

D) 

E) 

19. What is the smallest positive integer that is *not* a factor of 32! and is *not* a prime number?
Note: See problem 5 for the definition of the factorial function.

20. Seven students, Amy, Bob, Chris, David, Elisa, Frank, and Grace, are to give presentations in seven consecutive time slots. Chris will present immediately before David. Grace will present sometime after Chris. There will be two students who present in between Amy and Elisa. If Amy will present second, then who will be the third presenter?

A) Bob B) Chris C) David D) Frank E) Grace

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Math League Contest ~ Spring 2024 ~ Solutions

1.	84% of x is 65% of y gives: ① $0.84x = 0.65y$, and 91% of y is 56% of z gives: ② $0.91y = 0.56z$. Solving ① for y gives: $y = \frac{84}{65}x$, and solving ② for y gives: $y = \frac{56}{91}z$. Thus, $\frac{84}{65}x = \frac{56}{91}z \Rightarrow x = \frac{65}{84}\cdot\frac{56}{91}z \Rightarrow x = \frac{5\cdot 13}{3\cdot 4\cdot 7}\cdot\frac{2\cdot 4\cdot 7}{7\cdot 13}z \Rightarrow x = \frac{5}{3\cdot 7}\cdot\frac{2}{1}z \Rightarrow x = \frac{10}{21}z.$ Answer: A
2.	Letting $x = 1$ gives: $2024(-1)^4 - 2023(0)^3 + 2022(1)^2 - 2021(2) + 2020 = a + b + c + d + e.$ Which gives: $a + b + c + d + e = 2024 - 0 + 2022 - 4042 + 2020 = 2024$ Answer: 2024
3.	Since 31 (days in January) divided by 7 (days in a week) has a remainder of 3, there will be 3 consecutive days of the week that occur 5 times. There are 2 days between Saturday and Tuesday, but 3 days between Tuesday and Saturday, so there must be 5 Wednesdays, Thursdays and Fridays. Thus, January 1 must be on a Wednesday. Answer: Wednesday Alternate Solution: If January 1 were on a Sunday, Monday, or Tuesday, then there would be 5 Tuesdays in the month (since there are 31 days in January). If January 1 were on a Thursday, Friday, or Saturday, then there would be 5 Saturdays in the month. Thus, January 1 can only be on a Wednesday in this case.
4.	Since all these values are positive, raising them each to the same positive power preserves their relative size ordering. The LCM of 4, 6, and 9 is 36. Raising them each to the 36th power gives: $\left(2^{1/4}\right)^{36} = 2^9 = 512$, $\left(3^{1/6}\right)^{36} = 3^6 = 729$, and $\left(5^{1/9}\right)^{36} = 5^4 = 625$. Thus, $\sqrt[4]{2} < \sqrt[9]{5} < \sqrt[6]{3}$. Answer: B
5.	$\frac{(n+2)!}{(n-2)!} = \frac{(n+2)\cdot(n+1)\cdot n\cdot(n-1)\cdot(n-2)!}{(n-2)!} = (n+2)\cdot(n+1)\cdot n\cdot(n-1) = 421200$. Thus, the product of four consecutive positive integers is 421200. Since $20^4 = 160000$ and $30^4 = 810000$, the four integers must be in the 20's. We just need to express 421200 as the product of four integers in the 20's. $421200 = 100\cdot 4212 = 4\cdot 25\cdot 6\cdot 702 = (4\cdot 6)\cdot 25\cdot 9\cdot 78 = 24\cdot 25\cdot 9\cdot 3\cdot 26 = 24\cdot 25\cdot 26\cdot (3\cdot 9).$ Thus, $421200 = 24\cdot 25\cdot 26\cdot 27$, making $n = 25$. Answer: 25
6.	$100^x + 100^{1-x} = 25 \Rightarrow 100^x + 100\cdot 100^{-x} = 25 \Rightarrow 100^x + \frac{100}{100^x} = 25 \Rightarrow (100^x)^2 - 25(100^x) + 100 = 0.$ The LHS of the last equation factors: $(100^x - 5)(100^x - 20) = 0.$ Which gives: $100^x - 5 = 0$ or $100^x - 20 = 0$, solving for x and calling the smaller answer x_1 and the larger x_2, yields: $x_1 = \frac{1}{2}\log(5)$ and $x_2 = \frac{1}{2}\log(20)$. Thus, $\frac{x_2}{x_1} = \frac{\frac{1}{2}\log(20)}{\frac{1}{2}\log(5)} = \frac{\log(20)}{\log(5)} = \log_5(20).$ Answer: D

7. $\sqrt{\sin^4(x)+4\cos^2(x)} = \sqrt{\left(\sin^2(x)\right)^2+4\cos^2(x)} = \sqrt{\left(1-\cos^2(x)\right)^2+4\cos^2(x)}$

$= \sqrt{1-2\cos^2(x)+\cos^4(x)+4\cos^2(x)} = \sqrt{1+2\cos^2(x)+\cos^4(x)} = \sqrt{\left(1+\cos^2(x)\right)^2}$

$= \left|1+\cos^2(x)\right| = 1+\cos^2(x) = 1+1-\sin^2(x) = 2-\sin^2(x)$ Answer: D

Alternate (Shorter) Solution:

$\sqrt{\sin^4(x)+4\cos^2(x)} = \sqrt{\sin^4(x)+4\left(1-\sin^2(x)\right)} = \sqrt{\sin^4(x)-4\sin^2(x)+4} = \sqrt{\left(\sin^2(x)-2\right)^2}$

$= \sqrt{\left(2-\sin^2(x)\right)^2} = \left|2-\sin^2(x)\right| = 2-\sin^2(x)$

8. $\log_{2023}(x)\cdot\log_{2024}(x)=1 \Rightarrow \dfrac{\log_{2023}(x)}{\log_{2023}(2023)}\cdot\dfrac{\log_{2023}(x)}{\log_{2023}(2024)}=1 \Rightarrow \dfrac{\log^2_{2023}(x)}{\log_{2023}(2024)}=1$

$\Rightarrow \log^2_{2023}(x)=\log_{2023}(2024) \Rightarrow \log_{2023}(x)=\pm\sqrt{\log_{2023}(2024)} \Rightarrow x=2023^{\pm\sqrt{\log_{2023}(2024)}}$

x_1 is the smaller solution, which is $2023^{-\sqrt{\log_{2023}(2024)}} = \dfrac{1}{2023^{\sqrt{\log_{2023}(2024)}}}$. This answer is clearly positive and less than 1 ($\log_{2023}(2024)$ is slightly greater than 1). Thus, $0<x_1<1$ making $n=0$.

Answer: 0

9. Letting B represent the number of boys in class, the probability the first student selected is a girl is $\dfrac{10}{10+B}$, with $\dfrac{9}{9+B}$ being the probability the second student chosen is also a girl. The probability of both occurring is the product $\dfrac{10}{10+B}\cdot\dfrac{9}{9+B}$ which equals $\dfrac{3}{7}$. We need only solve for B:

$\dfrac{10}{10+B}\cdot\dfrac{9}{9+B}=\dfrac{3}{7} \Rightarrow \dfrac{90}{90+19B+B^2}=\dfrac{3}{7} \Rightarrow \dfrac{90}{90+19B+B^2}=\dfrac{90}{210} \Rightarrow 90+19B+B^2=210$

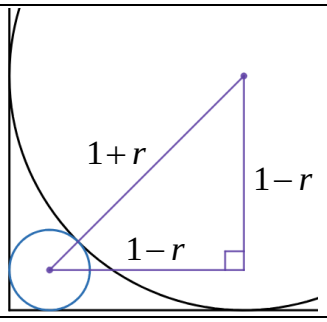
$\Rightarrow B^2+19B-120=0 \Rightarrow (B-5)(B+24)=0 \Rightarrow B=5$, taking the positive solution. Answer: 5

10. Letting $r=$ the radius of each of the four small circles and using the circle in the lower left corner as shown, the Pythagorean theorem gives:

$(1-r)^2+(1-r)^2=(1+r)^2 \Rightarrow 2-4r+2r^2=1+2r+r^2$

$\Rightarrow r^2-6r+1=0 \Rightarrow r=\dfrac{6\pm\sqrt{32}}{2}=3\pm2\sqrt{2}$

However, certainly $r<1$, which tells us $r=3-2\sqrt{2}$. Answer: C



11. $\log_{\frac{1}{2}}(x)+\log_2(x)+\log_4(x)+\log_8(x)=1$ can be rewritten using the change of base formula:

$\dfrac{\log_2(x)}{\log_2(1/2)}+\log_2(x)+\dfrac{\log_2(x)}{\log_2(4)}+\dfrac{\log_2(x)}{\log_2(8)}=1 \Rightarrow \dfrac{\log_2(x)}{-1}+\log_2(x)+\dfrac{\log_2(x)}{2}+\dfrac{\log_2(x)}{3}=1$

$\Rightarrow -\log_2(x)+\log_2(x)+\dfrac{1}{2}\log_2(x)+\dfrac{1}{3}\log_2(x)=1 \Rightarrow \dfrac{5}{6}\log_2(x)=1 \Rightarrow \log_2(x)=\dfrac{6}{5}$

Thus, $x=2^{6/5}$.

Answer: $2^{6/5}$

12. $i+2i^2+3i^3+4i^4+\ldots+2024i^{2024}=i-2-3i+4+5i-6+\ldots-2023i+2024$

$=(\underbrace{-2+4-6+8-10+12-\ldots-2022+2024}_{1012\text{ terms}})+(\underbrace{1-3+5-7+9-11+\ldots+2021-2023}_{1012\text{ terms}})i$

$=(\underbrace{2+2+2+\ldots+2}_{506\text{ terms}})+(\underbrace{-2-2-2-\ldots-2}_{506\text{ terms}})i=1012-1012i$

Answer: $1012-1012i$

13. Replacing x with the inverse function of $\frac{x+1}{x+2}$, which is $\frac{2x-1}{1-x}$, into the function yields:

$$f\left(\frac{\left(\frac{2x-1}{1-x}\right)+1}{\left(\frac{2x-1}{1-x}\right)+2}\right)=\frac{\left(\frac{2x-1}{1-x}\right)}{3\left(\frac{2x-1}{1-x}\right)+4}\Rightarrow f(x)=\frac{2x-1}{3(2x-1)+4(1-x)}=\frac{2x-1}{2x+1}.$$ Thus, $f(x)=\frac{2x-1}{2x+1}$ and so $f(1)=\frac{2-1}{2+1}=\frac{1}{3}$.

Answer: C

Alternate Solution: Taking the limit as $x\to\infty$ gives: $\lim_{x\to\infty} f\left(\frac{x+1}{x+2}\right)=\lim_{x\to\infty}\frac{x}{3x+4}\Rightarrow f(1)=\frac{1}{3}$.

14. $x^2+4x+1=0\Rightarrow x^2+1=-4x$, now divide by x to get: $x+\frac{1}{x}=-4$. Cubing yields:

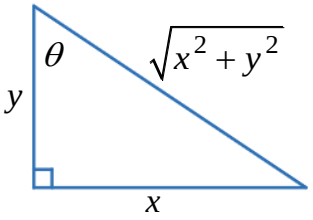
$$\left(x+\frac{1}{x}\right)^3=(-4)^3\Rightarrow\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=-64\Rightarrow\left(x^2+2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)=-64$$

$$\Rightarrow x^3+\left(x+2x+\frac{2}{x}+\frac{1}{x}\right)+\frac{1}{x^3}=-64\Rightarrow x^3+3\left(x+\frac{1}{x}\right)+\frac{1}{x^3}=-64\Rightarrow x^3+3(-4)+\frac{1}{x^3}=-64$$

$$\Rightarrow x^3-12+\frac{1}{x^3}=-64\Rightarrow x^3+\frac{1}{x^3}=-52$$

Answer: -52

15. $\frac{\sin(\theta)}{x}=\frac{\cos(\theta)}{y}\Rightarrow\frac{\sin(\theta)}{\cos(\theta)}=\frac{x}{y}\Rightarrow\tan(\theta)=\frac{x}{y}$, giving the right triangle:



From which we can obtain the sine and cosine values:

$\sin(\theta)=\frac{x}{\sqrt{x^2+y^2}}$ and $\cos(\theta)=\frac{y}{\sqrt{x^2+y^2}}$.

Thus, $\sin^2(\theta)-\cos^2(\theta)=\frac{x^2}{x^2+y^2}-\frac{y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}$.

Answer: A

16. $x+e^x=2024\Rightarrow e^{x+e^x}=e^{2024}\Rightarrow e^x e^{e^x}=e^{2024}$, letting $y=e^x$ gives: $ye^y=e^{2024}$, now apply the *Lambert W* function to get: $y=W\left(e^{2024}\right)\Rightarrow e^x=W\left(e^{2024}\right)$ and putting this into the original equation yields: $x+W\left(e^{2024}\right)=2024\Rightarrow x=2024-W\left(e^{2024}\right)$.

Answer: B

Alternate Solution:

$x+e^x=2024\Rightarrow e^x=2024-x\Rightarrow 1=(2024-x)e^{-x}$, now multiply the last equation by e^{2024} to get: $e^{2024}=(2024-x)e^{2024-x}$, then apply the *Lambert W* function to obtain: $W\left(e^{2024}\right)=2024-x$, which gives $x=2024-W\left(e^{2024}\right)$.

17. Let a, b, c, and d be positive integers.

I. $\sqrt{x}$ is rational $\Rightarrow \sqrt{x} = \frac{a}{b} \Rightarrow x = \frac{a^2}{b^2}$ is rational. ✓

II. x^2 and x^3 are rational $\Rightarrow x^2 = \frac{a}{b}$ and $x^3 = \frac{c}{d} \Rightarrow \frac{x^3}{x^2} = \frac{c/d}{a/b} \Rightarrow x = \frac{b \cdot c}{a \cdot d}$ is rational. ✓

III. Counterexample: $x = \sqrt{2}$ is irrational, but $x^4 = 4$ and $x^6 = 8$ are rational. ✗

IV. x^{-1} is rational $\Rightarrow x^{-1} = \frac{a}{b} \Rightarrow x = \frac{b}{a}$ is rational. ✓

Answer: E

18. $\arcsin(\sin(x)) = x$ only for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, since $-\frac{\pi}{2} \le \arcsin(x) \le \frac{\pi}{2}$, and for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ $\arcsin(\sin(x)) = \pi - x$. These in conjunction with $\sin(x)$ having a period of 2π makes graph E the only reasonable choice.

Answer: E

19. The smallest non-prime integer that is not a factor of $32!$, which equals $32 \cdot 31 \cdot 30 \cdot 29 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$, must have a prime factor greater than 32. The smallest prime number greater than 32 is 37, but primes are not allowed. The smallest non-prime that is not a factor of 32! must be $2 \cdot 37 = 74$.

Answer: 74

20. We know that A (Amy) is the second presenter. Thus, we can deduce that E (Elisa) must be in the 5th slot (since E cannot be before A).

Slot	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Presenter		A	C	D	E		

We know that C (Chris) and D (David) are consecutive with G (Grace) following them. Thus, the only way possible is if C and D take up the 3rd and 4th slots, respectively. However, G can be either in the 6th or 7th slot. But G's position has no bearing on C and D. Thus, C (Chris) must be 3rd.

Answer: B

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