New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2023

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. What is the limiting value of $\dfrac{\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{\cdots}}}}}}}{\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}}}}$, where the "$\cdots$" means that the pattern continues indefinitely?
   A) 1 B) $\sqrt{2}$ C) $\sqrt{3}$ D) 2 E) $\sqrt{6}$

2. If $\dfrac{x}{a-b}=\dfrac{y}{b-c}=\dfrac{z}{c-a}$, then what is the value of $x+y+z$?
   A) -1 B) 0 C) 1 D) $a+b+c$ E) $ab+ac+bc$

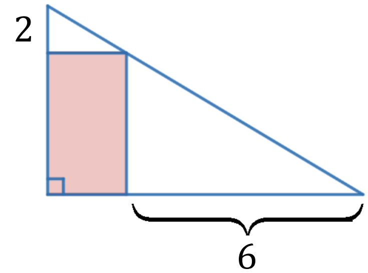
3. A customer buys an item from a street vendor for $25. The vendor selling the item makes no profit from this purchase. The customer gives the vendor a $100 bill, but the vendor does not have enough change. So, the vendor gets change from another street merchant. The vendor keeps $25 and gives the customer $75. Later, the street merchant comes to the vendor with the $100 bill which is stamped "COUNTERFEIT" and takes $100 back from the vendor. How much loss does the vendor incur?
   A) $25 B) $50 C) $75 D) $100 E) $125

4. The number 1234567891011121314…20222023 is formed by simply writing the counting numbers from 1 to 2023 in succession, creating a very large number. Counting from the left (i.e. starting with "1") which digit is in the 2023rd position?

5. What is the units digit of $1^3+2^3+3^3+4^3+\ldots+2022^3+2023^3$, i.e. what is the last digit of the sum of the cubes of all the integers from 1 through 2023?

6. How many unique (i.e. different) pairs of positive integers have 9 as their greatest common factor and 270 as their least common multiple?

7. The diagram, not drawn to scale, shows a right triangle with an inscribed rectangle (shaded). The height of the small triangle is 2, and the base of the larger triangle is 6, as shown. What is the area of the rectangle?



A) 8 B) 10 C) 12 D) $2\sqrt{40}$

E) There is not enough information to determine the area.

8. If $3^x = 32$ and $4^y = 27$, then what is xy?

A) $\log_7(27\cdot 32)$ B) 6.5 C) 7 D) 7.5 E) $\log_{12}\left(27^4\cdot 32^3\right)$

9. How many prime numbers are greater than $11!+1$, but less than $11!+12$?

Note: $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1$, for example $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.

A) 0 B) 1 C) 2 D) 3 E) 4

10. Suppose two points on the circumference of a circle are chosen at random. What is the probability that the length of the chord drawn between those two points is greater than the radius of the circle?

11. $\dfrac{1}{\log_9(6)} + \dfrac{1}{\log_{12}(6)} + \dfrac{1}{\log_{32}(6)} + \dfrac{1}{\log_{81}(6)} = ?$

A) $\dfrac{1}{\log_{134}(6)}$ B) 6 C) 7 D) 8 E) $\dfrac{4}{\log_{134}(6)}$

12. What is the maximum value of $x^2 + y^2$ with $x^2 + y^2 = 24x - 10y$, for real numbers x and y?

A) 26 B) 169 C) 338 D) 676

E) There is no maximum value, $x^2 + y^2$ is unbounded.

13. $\sin\left(2\cdot\arctan\left(\frac{\sqrt{2}}{2}\right)\right) = ?$

A) $\dfrac{1}{2}$ B) $\dfrac{\sqrt{6}}{3}$ C) $\dfrac{\sqrt{3}}{2}$ D) $\dfrac{2\sqrt{2}}{3}$ E) 1

14. If $x + y = 4$ and $x^3 + y^3 = 8$, then what is the value of $\dfrac{1}{x} + \dfrac{1}{y}$?

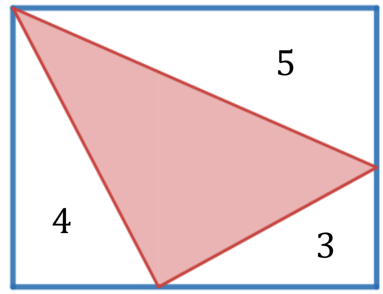
15. $x^{2+\log_2(x)} = 2$ has two solutions, call them x_1 and x_2, where $x_1 < x_2$. What is the value of $\dfrac{x_2}{x_1}$?

16. Let $f(x)=\sum_{n=1}^{2023}\cos(nx)=\cos(x)+\cos(2x)+\cos(3x)+\cdots+\cos(2023x)$. What is the period of $f(x)$?

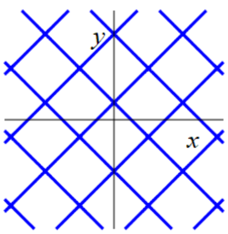
Recall: The period, P, of a function, $f(x)$, is the smallest value of $P>0$ for which $f(x+P)=f(x)$.

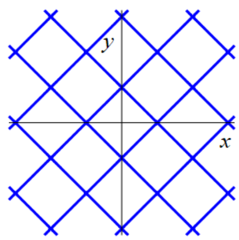
A) $\dfrac{\pi}{2023}$ B) $\dfrac{\pi}{2}$ C) π D) 2π E) 2023π

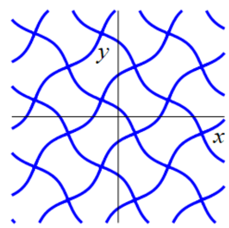
17. The diagram (not drawn to scale) shows a shaded triangle inscribed in a rectangle with one vertex at the upper left corner, another vertex somewhere along the bottom edge, and another somewhere along the right edge. The area of the three right triangles are given. What is the area of the inscribed (shaded) triangle?

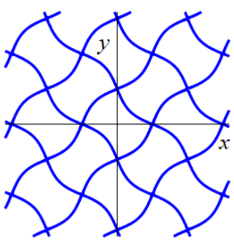


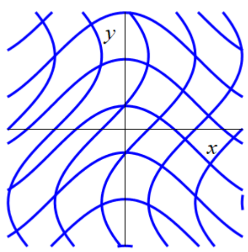
18. Which of the following graphs best represents the graph of $\sin(y)=\cos(x)$?

A) 

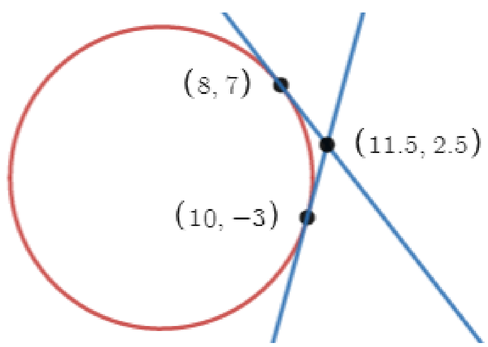
B) 

C) 

D) 

E) 

19. The accompanying diagram shows a circle with a tangent line at $(8,7)$ and another at $(10,-3)$. The two tangent lines intersect at $(11.5,\ 2.5)$. What is the radius of the circle?



20. Each of the cards below has a number on one side and a letter on the other side. Which card(s) must be turned over in order to verify that the following statement is true? *All cards with even numbers on one side must have a vowel on the other side.* The vowels are the letters: A, E, I, O and U.

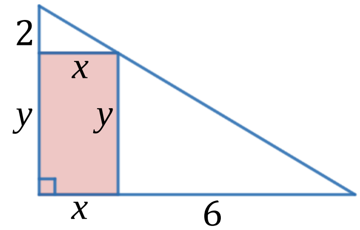
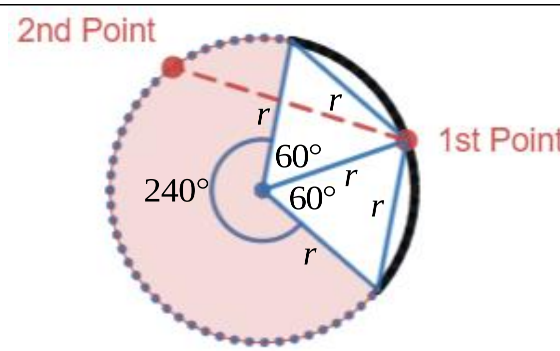
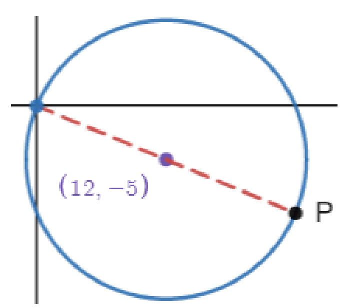
1: **A** 2: **7** 3: **8** 4: **X**

A) Only card 3. B) Only cards 1 and 3. C) Only cards 3 and 4.
D) Only cards 1, 3, and 4. E) All four cards.

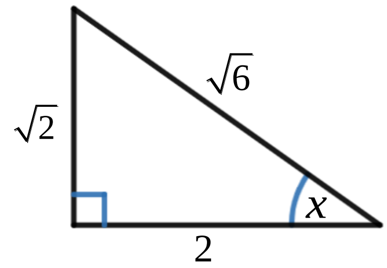
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2023 ~ Solutions

1.	For the numerator: let $x=\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{\cdots}}}}}}$, giving $x^2=6\sqrt{6\sqrt{6\sqrt{6\sqrt{6\sqrt{\cdots}}}}}=6x$. Thus, $x^2-6x=0 \Rightarrow x(x-6)=0 \Rightarrow x=0$ or $x=6$. Since x is clearly greater than 0, $x=6$. For the denominator: let $y=\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}}$, giving $y^2=6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}=y+6$. Thus, $y^2-y-6=0 \Rightarrow (y+2)(y-3)=0 \Rightarrow y=-2$ or $y=3$. Since y is clearly greater than 0, $y=3$. Hence, the limiting value is $\frac{x}{y}=\frac{6}{3}=2$. **Answer: D**
2.	Let $R=\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$. Thus, $R=\frac{x}{a-b}$, $R=\frac{y}{b-c}$, and $R=\frac{z}{c-a}$. These give: $x=R(a-b)=aR-bR$, $y=R(b-c)=bR-cR$, and $z=R(c-a)=cR-aR$. Therefore, $x+y+z=aR-Br+bR-cR+cR-aR=0$. **Answer: B**
3.	The vendor has incurred a loss of \$100, the amount of the counterfeit money. The customer no longer has the phony \$100 bill, nor does the other street merchant. Another way to see it is that the vendor kept the real \$25 change, but gave \$100 back to the street merchant, so the vendor has lost \$75 there. Additionally, the vendor lost the \$25 value of the item taken by the customer. **Answer: D**
4.	$\underbrace{123\ldots9}_{9\text{ digits}}\underbrace{101112\ldots99}_{2\cdot90=180\text{ digits}}\underbrace{100101102\ldots999}_{3\cdot900=2700\text{ digits}}1000\ldots2023$ Thus, the 2023rd digit falls among the 3-digit number sequence, it is the $2023-(9+180)=1834^{\text{th}}$ digit after the "99." $1834\div3=611$ with a remainder of 1. Hence, the digit we seek is the first digit after the first 611 3-digit numbers (after "99"). The 611th 3-digit number is $611+99=710$. The next digit after that is the "7" of 711. **Answer: 7**
5.	For $1^3+2^3+3^3+4^3+\ldots+2022^3+2023^3$, we need only keep track of the units digits. Let "$\rightarrow$" mean "only the units digit." Thus, the sum becomes: $\rightarrow\underbrace{1+8+7+4+5+6+3+2+9+0}_{10\text{ digit cycle}}\quad+\underbrace{\quad.\quad.\quad.\quad}_{219\text{ more cycles}}+\quad\underbrace{1+8+7}_{\text{from }2021^3+2022^3+2023^3}$ $=\underbrace{45+45+\ldots+45}_{220}+16\rightarrow0+6=6$. Hence, the units digit is "6." **Answer: 6**
6.	Letting m and n be the two numbers: $GCF(m,n)=9=3^2$ and $LCM(m,n)=270=2\cdot3^3\cdot5$. Thus, one of the numbers, m, must have 3^2 as a factor and the other, n, must have a factor of 3^3. Now we just need to "distribute" the 2 and 5. Here are the possibilities: ① $m=3^2\cdot2\cdot5$ and $n=3^3$, or ② $m=3^2\cdot2$ and $n=3^3\cdot5$, or ③ $m=3^2\cdot5$ and $n=3^3\cdot2$, or ④ $m=3^2$ and $n=3^3\cdot2\cdot5$. Thus, there are four different pairs. **Answer: 4**

7.	Let the base of the rectangle be x, and the height be y, as shown. The two triangles are similar, since they have corresponding angles. Thus, $\frac{2}{x}=\frac{y}{6} \quad\Rightarrow\quad xy=12$. Hence, the area is 12. **Answer: C** 
8.	$4^y=27 \Rightarrow (2^2)^y=3^3 \Rightarrow 3=2^{\frac{2}{3}y}$ Substituting this into the other equation gives: $(2^{\frac{2}{3}y})^x=32 \Rightarrow 2^{\frac{2}{3}xy}=2^5 \Rightarrow \frac{2}{3}xy=5 \Rightarrow xy=\frac{15}{2}$. **Answer: D**
9.	The integers between $11!+1$ and $11!+12$ are: $11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1+n$, where $n=2,3,4,5,6,7,8,9,10,11$. However, it is clear that for each of those n-values, n is a factor. Therefore, none of those integers can be prime. **Answer: A**
10.	The first point chosen can be anywhere on the circumference (shown as "1st Point" in the diagram). In order for the distance between the first and second points (the dashed line) to be greater than the radius, r, the second point must fall on the dotted arc that bounds the shaded region, which has an angle of $240°$. Thus, the probability the second point randomly falls on that arc is $\frac{240°}{360°}=\frac{2}{3}$. **Answer: $\frac{2}{3}$** 
11.	Using the change of base relation $\log_b(x)=\frac{\log_6(x)}{\log_6(b)}$, the expression can be written as: $\frac{1}{\left(\frac{\log_6(6)}{\log_6(9)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(12)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(32)}\right)}+\frac{1}{\left(\frac{\log_6(6)}{\log_6(81)}\right)}=\frac{\log_6(9)}{\log_6(6)}+\frac{\log_6(12)}{\log_6(6)}+\frac{\log_6(32)}{\log_6(6)}+\frac{\log_6(81)}{\log_6(6)}$ $=\frac{\log_6(3^2)}{1}+\frac{\log_6(2^2\cdot3)}{1}+\frac{\log_6(2^5)}{1}+\frac{\log_6(3^4)}{1}=\log_6(3^2\cdot2^2\cdot3\cdot2^5\cdot3^4)=\log_6(2^7\cdot3^7)$ $=\log_6(6^7)=7$. **Answer: C**
12.	$x^2+y^2=\left(\sqrt{x^2+y^2}\right)^2$, i.e. it is the square of the distance from $(0,0)$ to (x,y), where the point (x,y) is on the graph of $x^2+y^2=24x-10y$. Completing the square for x's and y's gives: $(x-12)^2+(y+5)^2=169=13^2$. Thus, the graph is a circle centered at $(12,-5)$ with radius 13. The point on the circle furthest from the origin will give the maximum value of x^2+y^2. The endpoints of the diameter on the line through origin are the points closest and furthest away from the origin. Since the circle passes through the origin, the furthest point is the other endpoint of the diameter, point P. The distance equals the diameter of the circle, which is 26. Hence, $x^2+y^2=26^2=676$. **Answer: D** 

13. Letting $x = \arctan\left(\frac{\sqrt{2}}{2}\right)$ gives the right triangle shown, since the tangent of an angle in a right triangle is $\frac{opposite}{adjacent}$. Using the double angle formula for sine, $\sin(2x) = 2\cos(x)\sin(x)$, and the triangle to get $\cos(x) = \frac{2}{\sqrt{6}}$ and $\sin(x) = \frac{\sqrt{2}}{\sqrt{6}}$, gives $\sin(2x) = 2 \cdot \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{2}}{\sqrt{6}}$. Which simplifies to $\frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$.



Answer: D

14. $x + y = 4 \;\Rightarrow\; (x+y)^3 = 4^3 \;\Rightarrow\; x^3 + 3x^2y + 3xy^2 + y^3 = 64 \;\Rightarrow\; x^3 + y^3 + 3xy(x+y) = 64$
$\Rightarrow\; 8 + 3xy \cdot 4 = 64 \;\Rightarrow\; 12xy = 56 \;\Rightarrow\; xy = \frac{14}{3} \;\Rightarrow\; \frac{x+y}{xy} = \frac{4}{\left(\frac{14}{3}\right)} \;\Rightarrow\; \frac{1}{y} + \frac{1}{x} = \frac{12}{14} = \frac{6}{7}$.

Answer: $\frac{6}{7}$

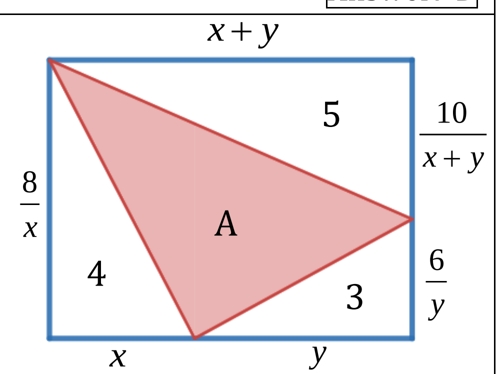
15. Taking $\log_2$ of both sides of $x^{2+\log_2(x)} = 2$ gives: $\left(2 + \log_2(x)\right)\log_2(x) = 1$. Now letting $y = \log_2(x)$ gives: $(2+y)y = 1 \;\Rightarrow\; y^2 + 2y - 1 = 0 \;\Rightarrow\; y = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$. Thus, $\log_2(x_1) = -1 - \sqrt{2}$ and $\log_2(x_2) = -1 + \sqrt{2} \;\Rightarrow\; x_1 = 2^{-1-\sqrt{2}}$ and $x_2 = 2^{-1+\sqrt{2}}$, making $\frac{x_2}{x_1} = \frac{2^{-1+\sqrt{2}}}{2^{-1-\sqrt{2}}} = 2^{2\sqrt{2}} = 4^{\sqrt{2}}$.

Answer: $2^{2\sqrt{2}} = 4^{\sqrt{2}}$

16. Since the period of $\cos(nx)$ is $\frac{2\pi}{n}$, the period of $\cos(x)$ is 2π, the period of $\cos(2x)$ is π, the period of $\cos(3x)$ is $\frac{2\pi}{3}$, and so on through the period of $\cos(2023x)$ being $\frac{2\pi}{2023}$. Hence, the period of the sum of all those periodic functions is the largest of the periods, i.e. the period of $\cos(x)$ which is 2π – since there is no smaller value of $P > 0$ for which $\cos(x+P) = \cos(x)$.

Answer: D

17. Let the base of the triangle with an area of 4 be x, making its height $\frac{8}{x}$; and the base of the triangle with an area of 3 be y, making its height $\frac{6}{y}$. Thus, the width of the triangle with an area of 5 is $x + y$, and its height is $\frac{10}{x+y}$. This gives: $\frac{10}{x+y} + \frac{6}{y} = \frac{8}{x}$. Multiplying the last equation by $xy(x+y)$ and simplifying results in: $3x^2 + 4xy - 4y^2 = 0$, which factors as $(3x - 2y)(x + 2y) = 0$. Solving for y gives: $y = \frac{3}{2}x$ or $y = -\frac{1}{2}x$, but both x and y must be positive, making $y = \frac{3}{2}x$. Hence, the area of the rectangle is $\frac{8}{x}(x+y) = \frac{8}{x}(x + \frac{3}{2}x) = \frac{8}{x} \cdot \frac{5}{2}x = 20$. We can finally obtain the area, A, of the shaded triangle: $A = 20 - (3 + 4 + 5) = 8$.

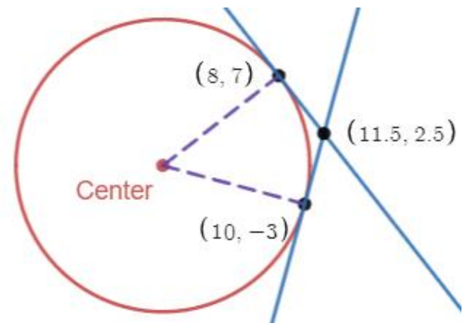


Answer: 8

18. Since $\cos(x)=\sin(x+\frac{\pi}{2})$ and they both have a period of 2π, then from $\sin(y)=\cos(x)$ we get $y=x+\frac{\pi}{2}+2\pi k$, for $k=0, \pm 1, \pm 2, \pm 3, \ldots$. These graphs are a family of parallel lines with slopes of 1. Also, we know $\cos(x)=\sin(\frac{\pi}{2}-x)$, so $\sin(y)=\cos(x)$ becomes $\sin(y)=\sin(\frac{\pi}{2}-x)$ together with the periodicity we finally obtain $y=\frac{\pi}{2}-x+2\pi k$, for $k=0, \pm 1, \pm 2, \pm 3, \ldots$. These graphs are a family of parallel lines with slopes of -1. Checking symmetry: replacing x with $-x$ gives: $\sin(y)=\cos(-x) \Leftrightarrow \sin(y)=\cos(x)$ – symmetric about the y-axis; replacing y with $-y$ gives: $\sin(-y)=\cos(x) \Leftrightarrow -\sin(y)=\cos(x) \not\Leftrightarrow \sin(y)=\cos(x)$ – not symmetric about the x-axis. Hence, graph (A).

Answer: A

19. The slope of the tangent line at $(8,7)$ is $\dfrac{7-2.5}{8-11.5}=\dfrac{4.5}{-3.5}=-\dfrac{9}{7}$, and the slope of the tangent line at $(10,-3)$ is $\dfrac{2.5-(-3.5)}{11.5-10}=\dfrac{5.5}{1.5}=\dfrac{11}{3}$. Since tangent lines to a circle are perpendicular to the radius at the point of tangency, the slope of the radius from the center to $(8,7)$ is $\dfrac{7}{9}$, and the slope of the radius from the center to $(10,-3)$ is $-\dfrac{3}{11}$. Thus, the equation of the line that contains the radius from the center to $(8,7)$ is $y-7=\frac{7}{9}(x-8) \Rightarrow y=\frac{7}{9}x+\frac{7}{9}$, and the equation of the line that contains the radius from the center to $(10,-3)$ is $y-(-3)=-\frac{3}{11}(x-10) \Rightarrow y=-\frac{3}{11}x-\frac{3}{11}$. The center is at the intersection of those two lines, which is at $(-1,0)$. Hence, the radius is the distance between $(-1,0)$ and either of the two points of tangency. Therefore, $r=\sqrt{(8-(-1))^2+(7-0)^2}=\sqrt{81+49}=\sqrt{130}$.



Answer: $\sqrt{130}$

20. Card 3 must be turned over to make sure there is a vowel on the other side. Card 4 must be checked as well, to make sure there is not an even number on the other side. The other cards need not be turned over, since no matter what is on the other side of them, the statement would not be violated.

Answer: C