



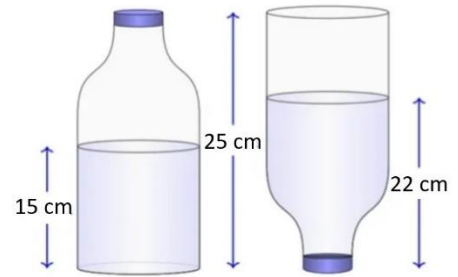
New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2026

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must be given in *exact* form, i.e. in terms of fractions, radicals, π , etc.

1. There are infinitely many integer pairs (m, n) that solve the equation $20m + 26n = 2026$, but only finitely many with both m and n positive. How many such pairs have only positive values? For example, the obvious solution $(100, 1)$ is one such pair.

2. A bottle with a sealed lid contains some water. The diagram shows the bottle upright and upside down along with the water levels and the height of the bottle. Assuming the bottle is cylindrical until it tapers from its shoulder up to its neck, which is above the water level in the upright bottle, what fraction of the bottle is occupied by water? Note: Water levels are not to scale.



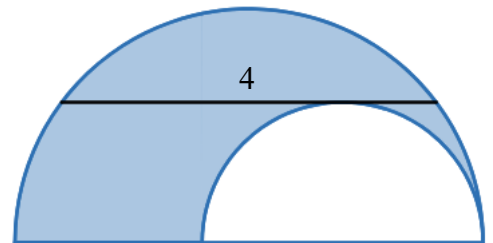
- A) $\frac{15}{22}$ B) $\frac{7}{10}$ C) $\frac{37}{50}$ D) $\frac{5}{6}$ E) $\frac{22}{25}$

3. Let $x = 2026! - 2025! + 2024! - \dots + 2! - 1!$, i.e. the signs alternate from $2026!$ down to $1!$. What is the remainder when x is divided by 10? Note: $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$, e.g. $3! = 3 \cdot 2 \cdot 1 = 6$.

4. If $x^x = 3$, then what is the value of $x^{x^{x+1}}$?

- A) $9\sqrt{3}$ B) e^3 C) 27 D) $27\ln(3)$ E) $27\sqrt[3]{3}$

5. A semicircle has a smaller semicircle drawn within it so its diameter lies along the diameter of the larger one. Then a line segment is drawn, with length 4, parallel to the diameters so it is tangent to the smaller semicircle and spans the larger semicircle, as shown. What is the area of the shaded region (i.e. the area between the two semicircles)?

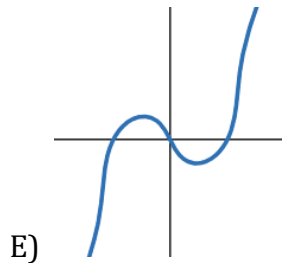
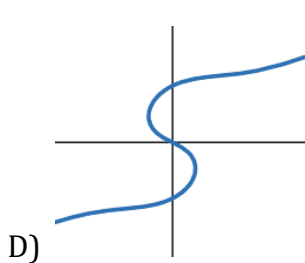
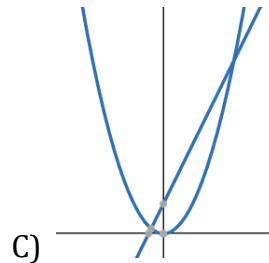
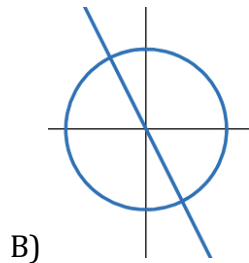
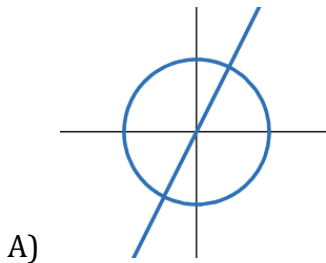


- A) π B) 2π C) 3π D) 4π
E) There is not enough information to make a determination.

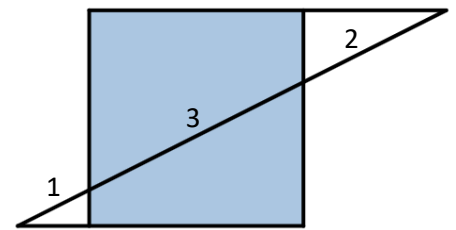
6. How many different seating arrangements are possible for five people in a row of five seats if a certain two people refuse to sit next to each other?
 A) 48 B) 60 C) 72 D) 84 E) 96

7. Suppose the polynomial $P(x) = ax^{2027} + bx^{2026} + cx^{2025} + 2026$ has $x^2 - 1$ as a factor. What is the value of b ?
 A) -2026 B) -1 C) 1 D) 2025 E) 2026

8. Which of the following graphs most closely depicts the graph of $2x^3 - x^2y + 2xy^2 - y^3 = 2x - y$?



9. A square has its top and bottom edges extended with a line segment connecting them so the hypotenuse of the lower left right-triangle that is formed has a length of 1 and the hypotenuse of the upper right right-triangle that is formed has a length of 2 and the part of the line segment within the square has a length of 3, as shown. What is the area of the square?



10. A *lattice point* is a point that has integer coordinates, for example: $(-1, -7)$, $(2, -3)$ and $(8, 1)$ are lattice points, while $(-1, \frac{8}{3})$ and $(1, 2.9)$ are not. Exactly how many lattice points satisfy the inequality $x^2 + y^2 \leq 10$?

11. What is the sum of the reciprocals of all solutions to the equation $|2x - |2x - 1|| = x$?

12. If $\log_x(y) + \log_y(x) = -2$, then what is xy ?

- A) $\frac{1}{e}$ B) $\frac{1}{2}$ C) 1 D) e E) There are infinitely many possible values.

13. A coin has been adjusted so the probability of it landing heads up is greater than $\frac{1}{2}$ and the probability that the first heads up occurring on the second toss is $\frac{4}{25}$. What is the adjusted probability of the coin landing heads up on one toss?

14. What point on the circle $(x-3)^2 + (y+4)^2 = 9$ maximizes $x^2 + y^2$?

15. The infinite sum $\sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{1}{10} + \frac{2}{100} + \frac{3}{1000} + \frac{4}{10000} + \dots$ converges to what value?

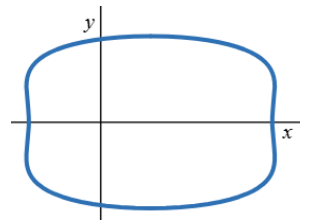
16. The equation $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$ has two real solutions. What is their product?

17. If $\cos(x) + \sin(x) = \frac{4}{3}$, then what is the value of $\sec(x) + \csc(x)$?

- A) $\frac{3}{4}$ B) 3 C) $\frac{16}{5}$ D) $\frac{24}{7}$ E) 4

18. Suppose you have a well-shuffled ordinary deck of 52 cards face down on a table. You then take a card from the top one at a time (and look at it) until you obtain an ace (i.e. an $A\spadesuit$, $A\heartsuit$, $A\clubsuit$, or $A\diamondsuit$). On average, how many cards must be turned over to obtain the first ace? Note: An ordinary deck of cards contains exactly four aces (the four indicated above).

19. The graph of the curve given by $y^4 - y^2 + x^2 - 2x = 5$ is shown to the right. What is its range?

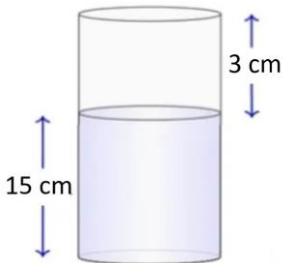
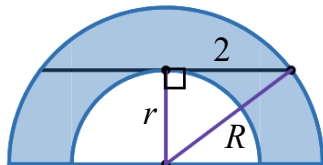


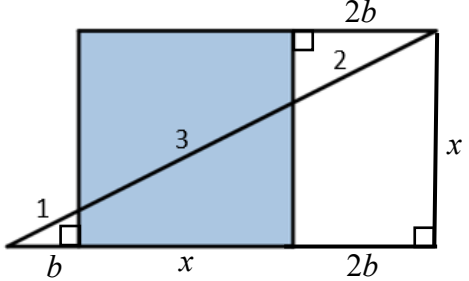
20. Three students in a logic class, Alice, Bob, and Cathy, are chatting and want to share their birthdays (month and day) by giving clues. Cathy gives Alice and Bob a list of ten possible dates for her birthday:

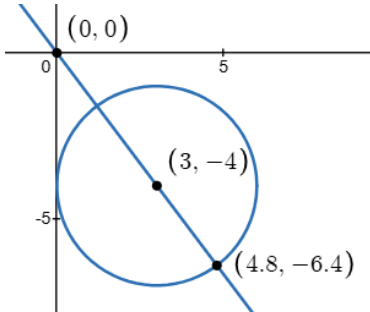
- March 15, 16, or 19 • April 17 or 18 • May 14 or 16 • June 14, 15, or 17

Cathy then tells Alice (but not Bob) the month, and tells Bob (but not Alice) the day. Alice then says, "I don't know Cathy's birthday, but I know that Bob doesn't know either." Bob then says, "At first I didn't know Cathy's birthday, but now I know!" Alice then says, "Now I know too!" Which of the ten dates is Cathy's birthday?

Math League Contest ~ Spring 2026 ~ Solutions

1.	$20m + 26n = 2026 \Rightarrow m = \frac{2026 - 26n}{20} = \frac{2020 + 6 - 20n - 6n}{20} = \frac{20(101 - n) + 6(1 - n)}{20}$ $\Rightarrow m = 101 - n + \frac{3}{10}(1 - n)$ Thus, $1 - n$ must be a multiple of 10 in order for m to be an integer. Solutions are: $n = 1, 11, 21, 31, 41, 51, 61, 71$, $n = 81$ is too large as it gives a negative value for m . Hence, there are only 8 ordered pairs with positive values. Answer: 8
2.	Clearly the volume of air in the upright bottle is the same as the volume of air in the upside-down bottle. Thus, we can imagine that we replace the non-uniform portion of the upright bottle that has air with the uniform cylindrical part of the upside-down bottle that also has only air, as shown. Now we see the water occupies 15 cm of the total 18 cm, for a fraction of $\frac{15}{18}$ which reduces to $\frac{5}{6}$. Answer: D 
3.	The remainder when a positive integer is divided by 10 is simply its units digit. Thus, we seek the units digit of x. All integers greater than or equal to 5 have a zero in the units position of their factorial ($5! = 120$, $6! = 720$, ...). Therefore, all we need to calculate is the units digit of $4! - 3! + 2! - 1!$. $4! - 3! + 2! - 1! = 24 - 6 + 2 - 1 = 19$. Answer: 9
4.	$x^{x^{x+1}} = x^{x^1 \cdot x^x} = x^{x^3} = (x^x)^3 = 3^3 = 27$ Answer: C
5.	We can move the smaller semicircle so its center coincides with the center of the larger semicircle without changing the areas. Letting the smaller semicircle have radius r and the larger one have radius R, we can now express the area of each: $A_{\text{smaller}} = \frac{1}{2}\pi r^2$ and $A_{\text{larger}} = \frac{1}{2}\pi R^2$. Thus, the area between the two semicircles is: $A_{\text{larger}} - A_{\text{smaller}} = \frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(R^2 - r^2)$. From the diagram we use the Pythagorean Theorem to obtain the relationship: $r^2 + 2^2 = R^2$ or $R^2 = r^2 + 4$. Now we can express the area we seek as: $A_{\text{larger}} - A_{\text{smaller}} = \frac{1}{2}\pi(r^2 + 4 - r^2) = 2\pi$. Answer: B 
6.	Rather than directly calculate the number of seating arrangements where the two people do not sit together, it will be easier to determine the number of arrangements where they <i>do</i> sit together, then subtract that from the total number of seating arrangements where there are no restrictions. There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ seating arrangements for 5 people without restrictions. For the number of seating arrangements where the two people must sit together, let's treat them as <i>one unit</i>. As such, we now have $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways they can be seated. However, the two people can be seated in two different ways (person A to the left of person B, or visa versa). Thus, there are $2 \cdot 24 = 48$ different seating arrangements for the 5 people if a certain two people <i>do</i> sit next to each other. Therefore, there are $120 - 48 = 72$ different seating arrangements. Answer: C

7.	Since $x^2 - 1$ is a factor of $P(x)$ and $x^2 - 1 = (x-1)(x+1)$, then both $x-1$ and $x+1$ must also be factors of $P(x)$. Thus, $x=1$ and $x=-1$ are roots of $P(x)$, i.e. $P(1)=0$ and $P(-1)=0$. $P(1)=0 \Rightarrow a(1)^{2027} + b(1)^{2026} + c(1)^{2025} + 2026 = 0 \Rightarrow$ ① $a+b+c+2026=0$ and $P(-1)=0 \Rightarrow a(-1)^{2027} + b(-1)^{2026} + c(-1)^{2025} + 2026 = 0 \Rightarrow$ ② $-a+b-c+2026=0$ Adding equations ① and ② gives $2b+4052=0 \Rightarrow b=-2026$ Answer: A
8.	$2x^3 - x^2y + 2xy^2 - y^3 = 2x - y \Rightarrow 2x^3 - x^2y + 2xy^2 - y^3 - 2x + y = 0$ Now factor by grouping. $\Rightarrow x^2(2x-y) + y^2(2x-y) - (2x-y) = 0 \Rightarrow (2x-y)(x^2 + y^2 - 1) = 0$ The only way the product of the two expressions can be zero is if: $2x-y=0$ or $x^2 + y^2 - 1 = 0$. $2x-y=0$ gives the line: $y=2x$, and $x^2 + y^2 - 1 = 0$ gives the circle: $x^2 + y^2 = 1$. Thus, the graph consists of both the line $y=2x$ and the circle $x^2 + y^2 = 1$. Answer: A
9.	Let's call the length of the sides of the square x, so we seek the value of x^2. The two small right triangles formed are similar. Thus, if we call the base of the small one b, the base of the larger one is $2b$. Then we extend the bottom edge to form a large right triangle whose base is $x+3b$, as shown. Applying the Pythagorean Theorem to the large right triangle gives ① $(x+3b)^2 + x^2 = (1+3+2)^2$. Using similar triangles again we get $\frac{x+b}{1+3} = \frac{b}{1} \Rightarrow x=3b$. Substituting that result into ① gives $(x+x)^2 + x^2 = 6^2 \Rightarrow 5x^2 = 36 \Rightarrow x^2 = \frac{36}{5} = 7.2$ Answer: $\frac{36}{5}$ or 7.2 
10.	Since $x^2 + y^2 = 10$ is the equation of a circle centered at the origin with a radius of $\sqrt{10} \approx 3.2$, we seek all the lattice points that are interior or on the circumference of the circle. Starting along the x-axis where $y=0$: $x=0, \pm 1, \pm 2, \pm 3$ for 7 points; for $y=1$: $x=0, \pm 1, \pm 2, \pm 3$ gives 7 points; for $y=2$: $x=0, \pm 1, \pm 2$ gives 5 points; $y=3$: $x=0, \pm 1$ gives 3 points. Thus, there are 15 points that satisfy the inequality that are above the x-axis, and by symmetry there are 15 more below the x-axis. Hence, there are a total of $7 + 2 \cdot 15 = 37$ lattice points that solve the inequality. Answer: 37
11.	We can break down $2x - 2x-1 = x$ into cases. Case I: If $2x-1 \geq 0 \Rightarrow 2x - (2x-1) = x \Rightarrow 1 = x \Rightarrow \boxed{x=1}$ Case II: If $2x-1 < 0 \Rightarrow 2x - (-(2x-1)) = x \Rightarrow 4x-1 = x$ Sub-Case A: If $4x-1 \geq 0 \Rightarrow 4x-1 = x \Rightarrow \boxed{x=\frac{1}{3}}$ Sub-Case B: If $4x-1 < 0 \Rightarrow -(4x-1) = x \Rightarrow -4x+1 = x \Rightarrow \boxed{x=\frac{1}{5}}$ Thus, the sum of the reciprocals is $1 + 3 + 5 = 9$. Answer: 9

12.	Let $w = \log_x(y) = \frac{\ln(y)}{\ln(x)}$ and $\log_y(x) = \frac{\ln(x)}{\ln(y)} = \frac{1}{w}$. Thus, the given equation can be written as $w + \frac{1}{w} = -2$, which can be written as $w^2 + 2w + 1 = 0$ or $(w+1)^2 = 0$. The only solution is $w = -1$, which tells us $\log_x(y) = -1$ or $y = x^{-1}$. Hence, $xy = 1$. Answer: C
13.	Let p be the probability of the coin landing heads up on a single toss. Thus, the probability that the first heads up occurs on the second toss is $(1-p) \cdot p$, since $1-p$ is the probability the coin lands tails up. But we are told this probability is $\frac{4}{25}$. Hence, we just need to solve $(1-p) \cdot p = \frac{4}{25}$ for p. $(1-p) \cdot p = \frac{4}{25} \Rightarrow 25p - 25p^2 = 4 \Rightarrow 25p^2 - 25p + 4 = 0 \Rightarrow (5p-1)(5p-4) = 0 \Rightarrow p = \frac{1}{5}$ or $p = \frac{4}{5}$. However, the problem states that this probability is greater than $\frac{1}{2}$, making $p = \frac{4}{5}$. Answer: $\frac{4}{5} = 0.8$
14.	Let (x, y) be a point on the circle $(x-3)^2 + (y+4)^2 = 9$. Since the distance from the origin to (x, y) is $\sqrt{x^2 + y^2}$, $x^2 + y^2$ is the square of that distance. The point on the circle that maximizes $x^2 + y^2$ is the point on the circle that maximizes the distance to the origin. The point we seek is the point where the circle intersects the line through the origin and the center of the circle that is furthest away from the origin, as shown. The line through the origin and the center of the circle is $y = -\frac{4}{3}x$. Now substitute $y = -\frac{4}{3}x$ into the equation for the circle: $(x-3)^2 + (-\frac{4}{3}x+4)^2 = 9$. $\Rightarrow x^2 - 6x + 9 + \frac{16}{9}x^2 - \frac{32}{3}x + 16 = 9 \Rightarrow \frac{25}{9}x^2 - \frac{50}{3}x + 16 = 0 \Rightarrow 25x^2 - 150x + 144 = 0$ $\Rightarrow (5x-24)(5x-6) = 0 \Rightarrow x = \frac{24}{5}$ or $x = \frac{6}{5}$ Taking the further point from $(0,0)$ gives $x = \frac{24}{5}$. Thus, $y = -\frac{4}{3} \cdot \frac{24}{5} = -\frac{32}{5}$. Hence, the point on the circle that maximizes $x^2 + y^2$ is $(\frac{24}{5}, -\frac{32}{5})$. Answer: $(\frac{24}{5}, -\frac{32}{5}) = (4.8, -6.4)$ 
15.	Let $x = \sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{1}{10} + \frac{2}{100} + \frac{3}{1000} + \frac{4}{10000} + \dots$, then $10x = \frac{10}{10} + \frac{20}{100} + \frac{30}{1000} + \frac{40}{10000} + \dots$, which simplifies to $10x = 1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \dots$. Subtracting the original equation from this gives: $10x - x = 1 + \left(\frac{2-1}{10}\right) + \left(\frac{3-2}{100}\right) + \left(\frac{4-3}{1000}\right) + \dots \Rightarrow 9x = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$ Now multiply by 9 to obtain a more recognizable RHS: $\Rightarrow 81x = 9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots = 9.999\dots$, which converges to 10. Thus, $81x = 10 \Rightarrow x = \frac{10}{81}$. Answer: $\frac{10}{81}$

16.	Cubing both sides: $\left((x+9)^{1/3} - (x-9)^{1/3}\right)^3 = 3^3 \Rightarrow (x+9) - 3(x+9)^{2/3}(x-9)^{1/3} + 3(x+9)^{1/3}(x-9)^{2/3} - (x-9) = 27$ $\Rightarrow x+9 - x+9 + 3(x+9)^{1/3}(x-9)^{1/3} \left(-(x+9)^{1/3} + (x-9)^{1/3}\right) = 27$ $\Rightarrow 18 - 3(x+9)^{1/3}(x-9)^{1/3} \underbrace{\left((x+9)^{1/3} - (x-9)^{1/3}\right)}_3 = 27 \Rightarrow -3((x+9)(x-9))^{1/3} \cdot 3 = 9$ $\Rightarrow (x^2 - 81)^{1/3} = -1 \Rightarrow x^2 = 80 \Rightarrow x = \pm\sqrt{80} \text{ Making the product } -80.$ Answer: -80
17.	Squaring both sides of the equation gives: $[\cos(x) + \sin(x)]^2 = \left(\frac{4}{3}\right)^2 \Rightarrow \cos^2(x) + 2\cos(x)\sin(x) + \sin^2(x) = \frac{16}{9}$ $\Rightarrow \cos^2(x) + \sin^2(x) + 2\cos(x)\sin(x) = \frac{16}{9} \Rightarrow 1 + 2\cos(x)\sin(x) = \frac{16}{9} \Rightarrow \cos(x)\sin(x) = \frac{7}{18}$ Now divide the left hand side of the given equation by $\cos(x)\sin(x)$ and the right hand side by $\frac{7}{18}$ to obtain: $\frac{\cos(x) + \sin(x)}{\cos(x)\sin(x)} = \frac{4/3}{7/18} \Rightarrow \frac{1}{\sin(x)} + \frac{1}{\cos(x)} = \frac{24}{7} \Rightarrow \csc(x) + \sec(x) = \frac{24}{7}.$ Answer: D
18.	On average, the 4 aces will partition the remaining 48 cards into 5 equal groups, i.e. $48/5 = 9.6$ non-aces in between pairs of aces. So, on average, the first 9.6 cards will be non-aces, then the next card will be an ace. Hence, on average the first ace will occur on the $9.6 + 1 = 10.6^{\text{th}}$ draw. Answer: 10.6
19.	$y^4 - y^2 + x^2 - 2x = 5 \Rightarrow x^2 - 2x + y^4 - y^2 - 5 = 0$ Using the quadratic formula to solve for x in terms of y gives: $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (y^4 - y^2 - 5)}}{2 \cdot 1} = \frac{2 \pm \sqrt{-4y^4 + 4y^2 + 24}}{2} = 1 \pm \sqrt{-y^4 + y^2 + 6}$ Thus, $-y^4 + y^2 + 6 \geq 0$ or $y^4 - y^2 - 6 \leq 0$. $y^4 - y^2 - 6 \leq 0 \Rightarrow (y^2 - 3)(y^2 + 2) \leq 0$. $\Rightarrow y^2 - 3 \leq 0 \Rightarrow -\sqrt{3} \leq y \leq \sqrt{3}.$ Answer: $-\sqrt{3} \leq y \leq \sqrt{3}$ or $y \in [-\sqrt{3}, \sqrt{3}]$
20.	In order for Alice to know that Bob doesn't know Cathy's birthday, the month she was told cannot be March or April. If the month were March or April, then it would be possible for Bob to know Cathy's birthday if the day was 18 or 19, since the 18th only occurs in April and the 19th only occurs in March. Thus, now Bob knows the month must be either in May or June. In order for Bob to now know Cathy's birthday, the day cannot be the 14th, since that day appears in May and June. Thus, Bob was told either 15, 16, or 17. In order for Alice to now know the correct birthday, the month must be May, for if it were June she would still not know (it could be June 15 or 17). Thus, the only logical birthdate is May 16. Answer: May 16